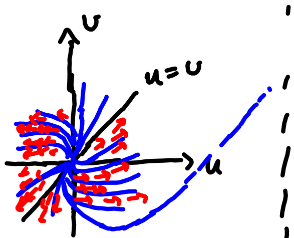
5.6 (2)

$\vec{X}' = \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}}_{P} \vec{X}$



Eigenvalues:

$0 = |\lambda I - P|$

$= \begin{vmatrix} \lambda-3 & 1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-3)(\lambda-1)+1$

$= \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$

$\lambda = 2$ has <u>multiplicity 2</u>

Eigenvectors: $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$

$2I - P = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

$(2I-P)\vec{u} = \vec{0} \iff a = b$

Eigenspace V spanned by

$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Defect $d = 2 - 1 = 1$

$=$ multiplicity $-$ dim V

Find $\vec{v}_2 \neq \vec{0}$ such that

$(P-2I)^2 \vec{v}_2 = \vec{0}$

$\vec{0} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\vec{v}_2$

Any $\vec{v}_2$ satisfies this.

Choose $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then $\vec{v}_2 \notin V$. Then

$(P-2I)\vec{v}_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{u}_1 \neq \vec{0}$

Let $\vec{v}_1 = \vec{u}_1$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

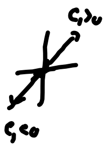
General solution:

$\vec{X}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$= e^{2t}\begin{bmatrix} c_1 + (1+t)c_2 \\ c_1 + t c_2 \end{bmatrix} = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \ t \in \mathbb{R}$

(c_1, c_2 arbitrary numbers)

$c_2 = 0$ gives $\vec{X}(t) = e^{2t}\begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$



so $u(t) = v(t)$, $t \in \mathbb{R}$.

Hence solutions curves are parts of the line $u = v$.

As $t \to -\infty$, $\vec{X}(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We have

$$\frac{u'(t)}{v'(t)} = \frac{e^{2t}[\,2(c_1+(t+1)c_2)+c_2\,]\frac{1}{t}}{e^{2t}[\,2(c_1+tc_2)+c_2\,]\cdot\frac{1}{t}} = \frac{2\left(\frac{c_1}{t}+(1+\frac{1}{t})c_2\right)+\frac{c_2}{t}}{2\left(\frac{c_1}{t}+c_2\right)+\frac{c_2}{t}} \xrightarrow[t\to\pm\infty]{} \frac{2c_2}{2c_2} = 1$$

5.6.3. $\vec{X}' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\vec{X} = P\vec{X}$,

$0 = |\lambda I - P| = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = \dots = (\lambda-3)^2$. $\lambda = 3$ with multiplic 2.

Eigenspace V:

$\vec{0} = (P-3I)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, $a+b=0$: dim $V = 1$,

V generated by $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; Defect <u>$d = 1$</u>

Find generalized eigenvector $\vec{v}_2$: $\vec{0} = (P-3I)^2\vec{v}_2 = [0]\vec{v}_2$.

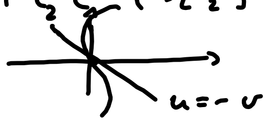
Let $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $(P-3I)\vec{v}_2 = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$= -2\vec{u}_1 \neq \vec{0}$

Let $\vec{v}_1 = -2\vec{u}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

General solution: $\vec{X}(t) = c_1 e^{3t}\begin{bmatrix} -2 \\ 2 \end{bmatrix} + c_2 e^{3t}\left(t\begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$

Here $\frac{u'(t)}{v'(t)} \xrightarrow[t\to\pm\infty]{} -1$



5.6 (7) $\vec{x}' = P\vec{x}$, $P = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

Eigenvalues:

$$0 = |\lambda I - P| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ -7 & \lambda-9 & -7 \\ 0 & 0 & \lambda-2 \end{vmatrix} = (\lambda-2)[(\lambda-2)(\lambda-9) - 0]$$

$= (\lambda-2)^2(\lambda-9)$

$\lambda = 9$: Multiplicity 1

$\lambda = 2$: —"— 2

$\lambda = 2$: $P - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -7 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

Eigenvectors $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ given by $a = b + c$.

Two dim. eigenspace V_2 spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{v}_1$ ($a=1, b=0, c=1$)

and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ($a=1, b=1, c=0$)

$\lambda = 9$: $P - 9I = \begin{bmatrix} -7 & 0 & 0 \\ -7 & 0 & 7 \\ 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ spans eigenspace V_9.

General solution:

$$\underline{\vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{9t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

5.6 (15) $\vec{x}' = P\vec{x}$, $P = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

Eigenvalues:

$$0 = |\lambda I - P| = \begin{vmatrix} \lambda+2 & 9 & 0 \\ -1 & \lambda-4 & 0 \\ -1 & -3 & \lambda-1 \end{vmatrix} = (\lambda-1)[(\lambda+2)(\lambda-4)+9]$$

$= (\lambda-1)(\lambda^2 - 2\lambda + 1) = (\lambda-1)^3$, $\underline{\lambda = 1 \text{ has multiplicity } 3}$.

Eigenvectors, $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$P - I = \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $a + 3b = 0$, c arbitrary. 2-dimensional eigenspace V_1 spanned by $\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ ($b=1, a=-3, c=0$) and

$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ($a = b = 0$, $c = 1$)

$\lambda = 1$ has defect $d = 1$. We find generalized eigenvector $\vec{u}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $(P - I)^2 \vec{u}_3 = \vec{0}$, or

$\vec{0} = \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Hence we may use any $\vec{u}_3 \notin V_1$, say $\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ($|\vec{v}_1\ \vec{v}_2\ \vec{u}_3| \neq 0$ here)

Moreover,

$\vec{u}_2 = (P - I)\vec{u}_3 = \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 3 \end{bmatrix}$ ($= 3\vec{v}_1 + 3\vec{v}_2$), $\neq \vec{0}$

Let $\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{v}_2$. Then $\vec{u}_1, \vec{u}_2, \vec{u}_3$ generate the solutions:

$$\underline{\vec{x}(t) = c_1 e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -9 \\ 3 \\ 3 \end{bmatrix} + c_3 e^t \left(t\begin{bmatrix} -9 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)}$$

5.6 (16) $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix}$, $\vec{X}' = P\vec{X}$:

Eigenvalues:

$$0 = |\lambda I - P| = \begin{vmatrix} \lambda-1 & 0 & 0 \\ 2 & \lambda+2 & 3 \\ -2 & -3 & \lambda-4 \end{vmatrix} = (\lambda-1)[(\lambda+2)(\lambda-4)+9]$$

$$= (\lambda-1)(\lambda^2-2\lambda+1) = \underline{(\lambda-1)^3}$$

$\lambda = 1$ has multiplic. 3.

Eigenvectors $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$: $P - I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 3 & 3 \end{bmatrix}$,

$2x + 3y + 3z = 0$. 2-dim. eigenspace V_1 , spanned by

$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $(x = 0, y = 1, z = -1)$, and

$\vec{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ $(x = 3, y = -2, z = 0)$

Defect $d = 1$. Find generalized eigenvector $\vec{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$ such that

$$\vec{0} = (P-I)^2\vec{v}_3 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix}\vec{v}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{v}_3 .$$

We may choose any $\vec{v}_3 \neq 0$, $\vec{v}_3 \notin V_1$, say $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\left(|\vec{u}_1\, \vec{u}_2\, \vec{v}_3| = \begin{vmatrix} 0 & 3 & 1 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{vmatrix} = -2 \neq 0 \right)$.

Here $(P-I)\vec{v}_3 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$ $(= -2\vec{u}_1)$ $\neq \vec{0}$.

Let $\vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$, $\vec{v}_1 = \vec{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$. Then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ generate the solutions:

$$\underline{\vec{X}(t) = c_1 e^t \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} + c_3 e^t \left(t\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)}$$

$(c_1, c_2, c_3$ arbitrary constants$)$.

5.6 (21) $P = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $\vec{X}' = P\vec{X}$:

Eigenvalues

$$0 = |\lambda I - P| = \begin{vmatrix} \lambda+1 & 4 & 0 & 0 \\ -1 & \lambda-3 & 0 & 0 \\ -1 & -2 & \lambda-1 & 0 \\ 0 & -1 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)(\lambda-1)[(\lambda+1)(\lambda-3)+4]$$

$$= (\lambda-1)^2(\lambda^2-2\lambda+1) = (\lambda-1)^4$$

$\lambda = 1$ has multiplicity 4.

Eigenspace V_1 : $\vec{u} = [a, b, c, d]^T$

$$P - I = \begin{bmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} +1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$a = 0$, $b = 0$, c and d arbitrary.

$\dim V_1 = 2$, V_1 spanned by

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Defect of $\lambda = 1$ is $4 - 2 = \underline{2}$. To find $\vec{v}_4 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that

$$\vec{0} = (P-I)^3 \vec{v}_4 = (P-I)^2 (P-I) \vec{v}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vec{v}_4$$

$= [0] \vec{v}_4$. Let $a = 1$, $b = c = d = 0$, then $\vec{v}_4 \notin V_1$.

Then

$$\vec{v}_3 = (P-I)\vec{v}_4 = \begin{bmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \neq \vec{0}$$

is another generalized eigenvector. Here

$$(P-I)\vec{v}_3 = \vec{v}_2 = (P-I)\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \vec{u}_2$$

Let $\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Get 4 lin. indep. sol.

$$\vec{x}_1(t) = e^t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_2(t) = e^t \vec{v}_2 = e^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\vec{x}_3(t) = e^t (t\vec{v}_2 + \vec{v}_3) = e^t \begin{bmatrix} -2 \\ 1 \\ 1 \\ t \end{bmatrix}$$

$$\underline{\vec{x}_4(t)} = e^t \left(\tfrac{1}{2} t^2 \vec{v}_2 + t \vec{v}_3 + \vec{v}_4 \right) = \underline{e^t \begin{bmatrix} -2t+1 \\ t \\ t \\ \tfrac{1}{2} t^2 \end{bmatrix}}$$