# *MAT 2440 spring 2017— Ark 1*

## *January 23, 2017*

This note is just a little complement to the book, underlining a few points, listing some terminology and giving examples and exercises (the same exercises as already posted on the web). There is also a very short recap of some technics you should know beforehand.

## *Terminology*

*ordinære differential ligninger*

Ordinary differential equations are the only type of differential equations we shall meet in this course. They just involve *one* independent variable in contrast to *partial differential equations* where there are several independent variables. One often thinks about the variable as *time* and denote it by $t$, but of course in many problems it will be some thing else. The acronym OED is frequently used.

The Euler–Bernoulli beam equation describes how a beam is deflected by external forces—like gravity—is an essential tool for engineers, and played a great role when the Eiffel tower was constructed:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2}{dx^2}w(x)\right) = q$$

is one where $x$ is the length measured along the beam.

*configuration space=konfigurasjonsrom*

The configuration space is the space where the sought for functions take their values. It can either be determined by assumption imposed on the sytem by the "real world"—that is, conditions on the sought for functions imposed by the application, or they can be confined to certain regions for mathematical reasons.

Frequently t he configuration space will be a part of the euclidean space $\mathbb{R}^n$. For instance the Lotka–Volterra equations

$$\begin{aligned}\dot{x} &= rx - axy\\ \dot{y} &= bxy - sy\end{aligned}$$

describing a prey-predator system, has the first quadrant as a configuration space, since the number of individuals must be positive. This is imposed by the biological assumptions; mathematically there is no reason to prohobit negative solutions.

A system with an angle as the dependent variable, like a body confined to move on a circle or an ellipse, has a circle as configuration space, and a rigid body in addition to moving around can rotate, has a configuration space equal to $\mathbb{R}^3 \times \mathrm{SO}(3)$.

THE PHASE SPACE is another space associated to a a system of differential equation. In our context this will mostly coincide with the configuration space. In physics, for instance, when one describes a set of moving particles, the configuration space consists of the possible positions of the particales, whereas the phase space consists of the possible positions and the possible velocities (or momenta). When formulating a system like in (1) beneath, the velocities are incorporated in the system as independent variables, so the distinction between the phase space and the configuration space space is not so clear.

*phase space=faserom*

*The group SO(3) describes the rotation in space and its element are the orthogonal . 3 × 3-matrices*

THE ORBIT or *integral curve* or *flow line* is the path a point in the phase space follows when exposed to the solution through that point. To be precise, assume that $x(t)$ is a solution of 1 such that $x(t_0) = x_0$ defined in an interval $J$ about $t_0$. Then the *orbit* of $x_0$ is the curve parametrized by $x(t)$.

*orbit=orbit eller bane; flow line=strømlinje.*

THE ORDER of a differential equation or a system of differential equations is the order of the highest derivative that occur in the equation. So among the two equations

*order=ordenen*

$$y' + p(t)y = q(t) \qquad y'' + p_1(t)y' + p_2(t)y + p_3(t) = 0$$

the first one is of order one whereas the second is of order two. Many differential equations arising in physics tend to be of the second order since the Newtons law $F = ma$ is of order two, but there are many examples of equations of higher order, for instance Euler–Bernoulli beam equation we mentioned above. Many other problems are modelled by first order equations like population dynamic and mixing problem arising in chemistry.

Every differential equation or system of differential equations can be recast as a system of first order equation to the price of increasing the number of equations in the system. To illustrate the procedure, consider the the $n$-th order equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = 0. \tag{$\ast$}$$

We shall transform this into a system of $n$ equations of the first order, and we do this by introducing the $n$ new functions $x_i$, for $0 \leq i \leq n-1$, by simply setting $x_i = y^{(i)}$. Notice that $x_0$ is just the original function $y$. In this way the equation ($\ast$) above becomes equivalent to the following linear system

$$\begin{aligned}
\dot{x}_0 &= x_1 \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{n-2} &= x_{n-1} \\
\dot{x}_{n-1} &= -\sum_j p_i x_i
\end{aligned}$$

For instance, a second order equation of the shape

$$\ddot{y} + p\dot{y} + qy = 0,$$

transforms into the system beneath that consists of two linear equations (remember $x_0$ is just another name for $y$).

$$\begin{aligned}
\dot{x}_0 &= x_1 \\
\dot{x}_1 &= \dot{x}_1 - qx_0
\end{aligned}$$

In matrix notation the system takes the form

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

The avantage of this seemingly futile manoeuvre is that it yields a unified way of formulating any system of ordinary differential, and concequently opens up the way to general methods to study such systems. Of course spesific methods for specific equations are usually the most effective, but they will always be based on the general theory.

Any system of ordinary differential equation can thus be brought on the form

$$\dot{x} = f(x, t) \tag{1}$$

where $x$ is vector whose components are continuously differentiable functions of a variable $t$ defined in an interval $I$ and taking values in $\mathbb{R}^n$ and the function $f(x, t)$ is continous and takes values in $\mathbb{R}^n$, and any initial value problem can be formulated as

$$\dot{x} = f(x, t) \qquad x(t_0) = x_0.$$

We are a little vague about where the function $f$ is de fiend, but it must be defined for $x$ in the configuration space and for $t$ at least in $I$.

AN AUTONOMOUS SYSTEM is one where the independent variable (*e.g.*, time) is on appearing explicitly in the equations, *i.e.*, the system is on the form

$$\dot{x} = f(x).$$

atonomous=autonomt

Loosely speaking, an autonomous system is subjected to externale driving forces.

A LINEAR linear sytem of differential equation is one where the sought for functions $x_1, \ldots, x_r$ and their derivatives enter *linearly*. Of two following examples the first one is linear, the second is not

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 \qquad & x' + x^2 + y = 0 \\ \dot{x}_2 &= x_1 + x_2 \qquad & y' + y^2 + 3x = 0 \end{aligned}$$

Nonlinear systems are usually very complicated to handle and they often show an unexpected behavior. Linear symestems are more bening, but of course, they can be challenging as well.

To be precise, let $f(u, t)$ be a function where the variable $u$ is a vector and the variable $t$ is a scalar. Assume that $F$ is *linear* in $u$ (but not necessarily in $t$). The differential equation we consider has the shape

$$\dot{x} = f(x, t) = 0. \qquad (*)$$

Using that differentiating functions is a linear operation one easily verifies that if $x_1$ and $x_2$ are solutions of (*) then any linear combination $c_1x_1 + c_2x_2$, with the $c_i$'s being constants, is a solution as well:

$$\begin{aligned} F(x_1 + c_2x_2, t) &= c_1F(x_1, t) + c_2F(x_2, t) = \\ &= c_1\dot{x}_1 + c_2\dot{x}_2 = d/dt(c_1x_1 + c_2x_2). \end{aligned}$$

The *superpositions* of to solutions of a *linear system* is a solution, or expressed slightly differently: the solutions of a linear differential equation form a vector space.

Be awear that our concept of linearity of a systems written on a form as in (1) corresponds to linear and homogenous equation in the traditional terminology. For example, the equation

$$y'' + py' + q = f(t)$$

is not *linear* in our terminology. It corresponds to the system

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p & -q \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

and the fuction to the right is not linear but *affine*.

OPPGAVE 1.1. Write the third order equation

$$y''' + py'' + qy' + ry = 0$$

on materix form. What is the determinant of the corresponding matrix? What is the characteristic polynomial? ✱

AN AFFINE SYSTEM or an affine differential equation is one of the type

affine=affin

$$\dot{x} = f(x,t) + g(t), \tag{2}$$

where $f$ is a continuos function linear in $x$ and defined for $t$ in some interval $I$, and $g\colon I \to \mathbb{R}^n$ is a continous function. The *assosiated linear system* is the system

$$\dot{x} = f(x,t).$$

OPPGAVE 1.2. Show that the general solution of of the affine system (2 ) can be written as the general solution of the associated linear system a particular solution of (2) (that mens just one). ✱

## *The logistic equation*

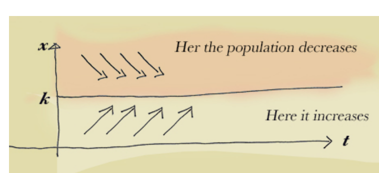

This equation is also called the *Verhulst equation* after Pierre François Verhulst It describes a population $x(t)$ by the differential equation

$$\dot{x} = rx(1 - x/k), \tag{3}$$

where $r$ is the *net growth rate* of the population when there are sufficiently resources. The net growth rate includes both birth rates and death rates, and it is one of the fundamental assumption on the model that this rate is proportionalto the population and independent of $x$. When there is a "fight for food" one assumes in the model that the growths rate declines with the factor $(1 - x/k)$ where $k$ is called the *carrying capacity* of the population. The carrying capacity $k$ is the maximal population that can be supported in the long run.

The *phase space*; that is, the space where $x$ can take values, equals $\mathbb{R}^+ = \{\, x \mid x > 0 \,\}$.

Without doing any calculations, one can understand some qualitative aspects of the solutions. Clearly $x(t) = k$ and $x(t) = 0$ are solutions, they are called the *equilibria* or the *equilibrium solutions*. If $x > k$, one sees that the derivative $\dot{x}$ is negative, and the population declines; however if $x < k$, then $\dot{x}$ is positive, and the population growths.




We can go one step further in the qualitative analysis and differentiate 3 to obtain

$$\ddot{x} = r(1 - 2x/k)\dot{x} = r^2 x(1 - x/k)(1 - 2x/k).$$

So we see that at points where $x = k/2$ the solution has an inflection point. If $x < k/2$, or $x > k$ the solution will be convex whereas it is concave when $k/2 < x < k$.

IT IS NOT DIFFICULT TO SOLVE the logistics equation explicitly by the method of separation of variables. To ease the calculation we replace $x$ by $x/k$, then it takes the form

$$\dot{x} = rx(1 - x).$$

Dividing by $x(1 - x)$ and decomposing in the partial fraction gives is

$$(\frac{1}{x} + \frac{1}{1-x})dx = rdt,$$

and integrating we arrive at

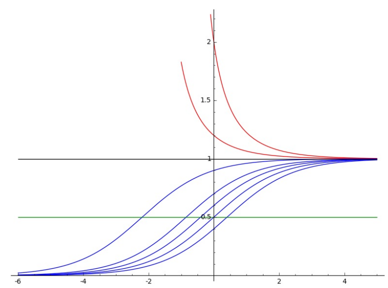
$$\log|x/(1 - x)| = rt + C$$

where $C = \log|x_0/(1 - x_0)|$. Hence it holds that

$$x/(1 - x) = x_0 e^{rt}/(1 - x_0)$$

and finally we find

$$x(t) = x_0/(x_0 + (1 - x_0)e^{-rt}).$$



After reinserting $x/k$ for $x$ and $kr$ for $r$ the complete solution is:

$$x(t) = x_0/(x_0 + (k - x_0)e^{-rt}).$$

OPPGAVE 1.3. If one also accepts negative solutions of the logistic equation, discuss the behavior when the initial value $x_0$ is negative. ✷

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OPPGAVE 1.4. There is a version of the logistic equation that incorporates a harvesting term $h$; that is, a constant term representing a continuos exploitation of the population. The equation the looks like

$$\dot{x} = rx(1 - x/k) - h.$$

Find the general solution and discussion the qualitative aspects. ✷

## *Two species— The Lotka–Volterra equation*

In cas the population sustem one studies has two species there will be two functions to determine, $x(t)$ and $y(t)$, and the phase space in this case will be the first quadrant in $\mathbb{R}^2$.

The first situation we look at is when to species both are "vegetarian", that is they are not eating each other. A simple model would then be

$$\dot{x} = x(r - ax - by)$$
$$\dot{y} = y(r' - a'y - b'x)$$

where $r$ and $s$ are the two growth rates for the two species, and the other coefficients represents the "fight for the resourse"; they are all assumed to be positive. The terms $-bxy$ and $-b'xy$ reflects the the fact that $y$-species and the $x$-sepspecies use the same resources, and that the propoersotanl ti the product $xy$ is one of the fundamental assumptions of the model.

## *Recap of some classical techniques*

### *First order linear equations*

These are equations with initial value problems of the form

$$y' + p(t)y = q(t) \qquad y(t_0) = y_0 \tag{4}$$

where $t$ is assumed to be in an interval $I$ where $p$ and $q$ are continuos functions. There is always a unique solution of such a problem, and there is a procedure to determine the solution expressed as integrals—may be one can explicitly compute those integrals or many can not, anyhow integrals are nice to handle. The trick is to use *an integrating factor*. So let $F$ be a primitive function for $p$ vanishing at $t_0$; that is a function such that $F' = p$ and $F(t_0) = 0$. When multiplied by $e^{-F}$ equation (4) above becomes

This means that $F(t) = \int_{t_0}^{t} p(t)dt$.

$$(e^F y)' = e^F y' + pe^F y = qe^F,$$

and integrating once more, we arrive at

$$y(t) = e^{-F(t)}(\int_{t_0}^{t} q(t)e^{F(t)} + y_0)dt.$$

Oppgave 1.5. Determine the general solution of the equation

$$y' + y = \sin t.$$

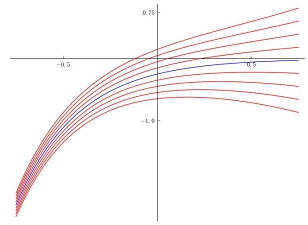
Determine a solution such that $y(0) = 2$, and one with $y'(3\pi/2) = 1$.

✷

Oppgave 1.6. Determine the general solution of the equation

$$y' - y = ae^{-bt} \tag{5}$$

where $a$ and $b$ are positive constants. Show that $y_0(t) = -a/(b+1)e^{-bt}$ is a solution of the equation (5) and that it is the only solution that tends to a limit when $t \to \infty$. Assume that $y$ is a solution such that for some $t_0$ it holds that $y(t_0) > y_0(t_0)$ then $y(t) \to \infty$ then $\lim_{t\to\infty} y(t) = \infty$ whereas $\lim_{t\to\infty} y(t) = -\infty$ for solutions satisfying $y(t_0) < y_0(t_0)$. HINT: Answer: $ce^x - a/(b+1)e^{-bx}$. ✱




OPPGAVE 1.7. Find the general solution of

$$(1+x^2)y' + 3xy = 6x.$$

✱

OPPGAVE 1.8. Let $f(t)$ be a function defined and twice differentiable in an interval $I$ about zero, and at assume that $f$ is positive. Let $y(t)$ be a solution of the equation

$$f(t)y' + 3ty = 6t.$$

Show that $y$ has a local maximum at $t = 0$ if $y(0) > 2$ and a local minimum if $y(0) < 2$. ✱

## *Separable equations*

Some first order equations of particular forms can be solved, at least partially, even if they are not linear. One class of such are the separable equations. They are of the form

$$y'(t) = f(y)g(t)$$

where $f$ and $g$ are functions. Dividing through by $f(t)$ and integrating one obtains

$$\int \frac{dy}{g(y)} = \int f(t)dt.$$

And if one is capable of both evaluating the integrals and solving the ensuing equation for $y$, one has a solution. For example if $y' = xy$ one finds $y^{-1}dy = dx$ and hence upon integrating one arrives at $\log|y| = x^2/2 + c$ where $c$ is arbitrary. Solving for $y$, one finds the general solution $y = ce^{-x^2/2}$ where $c$ is an (other) arbitrary constant.

OPPGAVE 1.9. Find all solutions of

$$xy' = y.$$

Can you give a geometric explanation of the equation? ✱

OPPGAVE 1.10. Determine all solutions of the differential equation

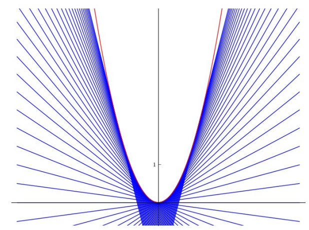
$$y' + \sqrt{y - c} = 0$$

where $c$ is constant. Show that for any $y_0 > c$ there infinitely many continuously differentiable solutions satisfying $y(t_0) = y_0$. How many are twice differentiable? ✷

OPPGAVE 1.11. Consider the equation

$$y = xy' - (y')^2, \tag{6}$$

which is one of the equations named *Clairot's equations*. If $y$ is solution of (6) show that $y' = x/2$ or $y'' = 0$, and conclude that the solutions are either $y = x^2/4$ or $y = ax - a^2$ where $a$ is an arbitrary constant. Show that the linear solutions are all the tangent to the parabola $y = x^2/4$. What condition on must $(x_0, y_0)$ satisfy for (6) to have a solution with $y(x_0) = y_0$? ✷



*Homogenuus equations*

These are equations of the form

$$y' = F(x, y)$$

where $F$ is a homogenous of $x$ and $y$ of degree zero; for example, $F$ can be the quotient of two homogenous polynomials of the same degree.

To solve such an equation, the trich is the substituion $y = ux$. Then $y' = u'x + u$, so we find

$$u'x + u = F(ux, x) = F(u, 1)$$

and this is a separable equations.

EKSEMPEL 1.1. Let the equation be

$$y' = \frac{x+y}{x-y}$$

. Setting $y = ux$, we find

$$u'x + u = \frac{x + ux}{x - ux} = \frac{1+u}{1-u}$$

which yields

$$\frac{(1-u)du}{(1+u^2)} = \frac{dx}{x}.$$

Integration gives the realtion

$$\arctan u - \log(1 + u^2) = \log|x| + C$$

that defiens $u$ implicityly. One can solve $u$ from this equation, but $x$ can be solved in terms of $u$

$$x = C(1+u^2)\exp(\arctan u).$$

❄

*Second order equations with constant coefficients*

These equations are of the form

$$y'' + ay' + by = 0 \qquad (✢)$$

where $a$ and $b$ are constants. To the equation (✢) one associates the following quadratic equation which is called the *characteristic equation*

$$r^2 + ar + b = 0, \qquad (✤)$$

and whose roots give the solutions of (✢). The salient point is to test if an exponential function $e^{rt}$—where $r$ very well can be a complex number—can satisfy (✢). Using that the derivative of an exponential is given as $(e^{rt})' = re^{rt}$, one easily finds

$$(e^{rt})'' + a(e^{rt})' + be^{rt} = (r^2 + ar + b)e^{rt},$$

hence the function $e^{rt}$ solves (✢) if and only if $r$ is a root of the characteristic equation (✤). This almost reduces the solution of (✢) to the solution of the cacac algebraic equation (✤),

IF $r_1$ AND $r_2$ ARE TWO DIFFERENT solutions of (✤), any linear combination of $e^{r_1t}$ and $e^{r_2t}$ solves (✢); that is, the general solution is the following

$$c_1e^{r_1t} + c_2e^{r_2t}.$$

THERE ARE BASICALLY THREE CASES when the coefficients $a$ and $b$ are real. Either the characteristic equation has two real roots, one double real root, or two complex conjugate roots. Which case occurs depends on the sign of the *discriminant* $\Delta = a^2 - 4b$ (that's the one under the square root in the formula).

In case the roots are real and different, the solutions are simply

$$c_1e^{r_1t} + c_2e^{r_2t}.$$

The asymptotic behavior of these functions depends on the signs of $r_1$ and $r_2$ (and of course on the constants $c_1$ and $c_2$). If both are negative they tend to zero as $t$ tends to infinity. If both roots are positive the solutions tend to $\pm\infty$; the sign depending on the sign of the dominating term; that is, the term corresponding to the greatest $r_i$. When

*Asymptotic behavior means the behavior when $t \to \infty$*

the signs of $r_1$ and $r_2$ are different, the solutions tends to $\pm\infty$ when $t$ tends to $\infty$ and $-\infty$, depending on the signs of the constants $c_i$.

In case the two roots are complex they must be complex conjugate since the characteristic equation has real coefficients. Hence we may write $\lambda_\pm = \alpha \pm \beta i$ with $\alpha$ and $\beta$ real. The corresponding complex constants are also conjugate so $c_\pm = u \pm vi$ where $u$ and $v$ are real vectors. We thus find, after some computation, the general solution

$$c_+ e^{\alpha t} e^{\beta i} + c_- e^{\alpha t} e^{-i\beta t} = e^{\alpha t}(u\cos\beta t + v\sin\beta t). \quad (7)$$

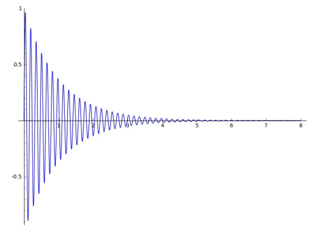


*Figure 1.1: A damped oscullation.*

In case $\alpha < 0$ the asymptotic behavior is a damped oscillation with circular frequency equal to $\beta$, if $\alpha > 0$ it is an "exploding oscillation". The case $\alpha = 0$ gives an oscillation with constant amplidude, and it is called a harmonic oscillation. The general solution from (7) may be brought on the form

$$x(t) = Ae^{\alpha t}\sin(\beta t + \phi)$$

where now the constants of integtation are the amplitude $A$ and the phase angle $\phi$.

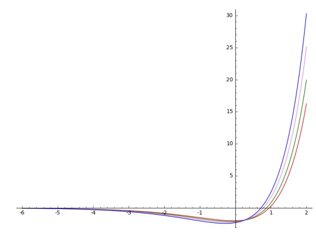


*Figure 1.2: Some integral curves for $y'' - 2y' + y = 0$ with the double root $r = 1$.*

The last case occures when the characteristic equation has a double root, say $r$. The general solution is on the form

$$x(t) = (c_1 + c_2 t)e^{rt}.$$

The Wronskian determinant is a useful tool when working with ODE's. It can be defined for any number of functions, but in our present context we stick to a pair of functions. The main reason the Wronskian is interesting is that it detects whether solutions of a linear equation are linearly independent or not. So let $y_1$ and $y_2$ be two functions. The *Wronskian* is defined as the determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

*Figure 1.3: Józef Maria Hoene-Wroński (1776–1853) Polish mathematician and Messianist philosopher*

Wronskian=Wronski-determinanten

Since any linear relation between the two functions persist between their derivatives, it is clear that the Wronskian of two linearly dependent functions vanishes, but the interesting property of $W(y_1, y_2)$ is that the converse holds as well provided $y_1$ and $y_2$ are solution of the same linear second homogenous equation. Indeed, Niels Henrik Abel gave the formula beneath for the Wronskian $W$

$$W(y_1, y_2) = K\exp(-\int_{x_0}^{x} a(t)dt) \quad (8)$$

where $y_1$ and $y_2$ are solutions of

$$y'' + a(t)y' + b(t)y = 0. \quad (\star)$$

and $x_0$ is any point in the interval where one considers the equation. Indeed, one finds upon derivation

$$W' = (y_1y_2' - y_2y_1')' = y_1y_2'' - y_2y_1'' = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}$$

Using that both functions $y_1$ and $y_2$ satisfy the equation (✬) one finds

$$\begin{aligned} W' = y_1y_2'' - & y_2y_1'' = \\ & = y_1\big(-a(t)y_1' - b(t)y_1\big) - y_2\big(-a(t)y_2' - b(t)y_2\big) = -a(t)W. \end{aligned}$$

that is $W$ satisfies the first order differential equation

$$W' = -aW$$

which has the solution as given in (8).

OPPGAVE 1.12. Show in detail that Wronskian of two linearly dependent functions vanishes. ✸