Exam MAT2440, spring 2017 — solutions

Problem 1

Let A be the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}.$$

Determine $\exp(tA)$.
Let $b(t)$ be given as

$$b(t) = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \end{pmatrix}.$$

Find the solution of the differential equation

$$\dot{x} = Ax + b$$

with $x(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

a) One finds that the characteristic polynomial of A equals $(\lambda + 1)^2$ so that -1 is an eigenvalue of multiplicity 3. Let $N = A + I$, then N is nilpotent and of course it commutes with I, and one has

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{pmatrix}$$

One finds

$$N^2 = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $N^3 = 0$. It follows that $\exp(tA) = \exp t(-I + N) = \exp(-tI)\exp(tN)$ and hence

$$\exp tA = e^{-t} \exp tN = e^{-t}(I + tN + t/2 \cdot N^2) = e^{-t} \begin{pmatrix} 1 & -1/2 \cdot t^2 & t^2 + t \\ 0 & 1 - 2t & 4t \\ 0 & -t & 1 + 2t \end{pmatrix}$$

b) Using $\exp -tA$ as an integrating factor brings the equation on the form

$$d/dt\big(e^{-tA}x(t)\big) = e^{-At}b(t).$$

Computing $e^{-tA}b(t)$ gives

$$e^{-At}b(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Integrating and taking care of the initial value gives

$$e^{-tA}x(t) = \begin{pmatrix} t+1 \\ 2 \\ 1 \end{pmatrix}.$$

Finally, one finds

$$x(t) = e^{tA}\begin{pmatrix} t+1 \\ 2 \\ 1 \end{pmatrix} = e^{-t}\begin{pmatrix} 2t+1 \\ 2 \\ 1 \end{pmatrix}.$$

Problem 2

Consider the optimalization problem

$$J(x) = \int_0^1 (2x^2 + 2x\dot{x} + \dot{x}^2)dt \qquad x(0) = 0 \quad x(1) = 1.$$

Determine the corresponding Euler-Lagrange equation and find the general solution of it.
Determine a function $x(t)$ with $x(0) = 0$ and $x(1) = 1$ such that $J(x)$ is minimal.

a) The integrand in $J(x)$ is derived from the function $L(q, p) = 2q^2+2qp+p^2$. One finds $L_q = 4q + 2p$ and $L_p = 2q + 2p$. Hence the EL-equation becomes

$$L_q(x, \dot{x}) - d/dt\big(L_p(x, \dot{x})\big) = 4x + 2\dot{x} - d/dt(2x + 2\dot{x}) = 4x - 2\ddot{x} = 0$$

or

$$\ddot{x} = 2x.$$

The general solution of the EL-equation is $x(t) = Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t}$.

b) The condition $x(0) = 0$ gives $B = -A$ and $x(1) = 1$ gives $Ae^{\sqrt{2}} + Be^{\sqrt{2}} = 1$, that is $A(e^{\sqrt{2}} - e^{-\sqrt{2}}) = 1$; that is $A = (e^{\sqrt{2}} - e^{-\sqrt{2}})^{-1}$. It follows that

$$x(t) = \frac{e^{\sqrt{2}t} - e^{-\sqrt{2}t}}{e^{\sqrt{2}} - e^{-\sqrt{2}}}.$$

This is a minimum since L is convex; indeed, $L_{qq} = 4$, $L_{pq} = 2$ and $L_{pp} = 2$ which gives the Hessian

$$\det \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} = 4$$

which is positive, and as L_{pp} and L_{qq} are positive as well, L is convex.

Problem 3

Let $\alpha > 0$. Show that $f(x) = x^{1/2}$ is Lipschitz in the interval $[\alpha, \infty)$ and give a Lipschitz constant. Is $f(x)$ Lipschitz in any interval of type $[0, \beta]$?
Consider the differential equation

$$\dot{x} = 2x^{1/2} \tag{*}$$

in the region $E = \{ x \in \mathbb{R} \mid x > 0 \}$. Determine the general solution of the equation and for each $t_0 \in \mathbb{R}$ and each $x_0 \in E$ a solution with $x(t_0) = x_0$. Give an argument for why it is unique.
Let now $F = \{ x \in \mathbb{R} \mid x \geq 0 \}$. Show that for every t_0 there are at least two solutions of (*) in F with $x(t_0) = 0$ and that both are determined for all t.

a) One has $f'(x) = 1/2 \cdot x^{-1/2}$ which is decreasing for $x > 0$. Our function f is C^1 on intervals $< x, y > \subseteq < \alpha, \infty >$ with $0 < \alpha$. The Mean Value Theorem gives

$$|f(x) - f(y)| = 1/2 \cdot c^{-1/2} \, |x - y| \leq 1/2 \cdot \alpha^{-1/2} \, |x - y|$$

and f is Lipschitz in $< \alpha, \infty >$ with a Lipschitz constant $1/2 \cdot \alpha^{-1/2}$. The function f is not Lipschitz in $[0, \beta]$, since again by the Mean Value Theotem one has

$$f(x) = 1/2 \cdot c^{-1/2} x$$

where $0 < c < x$ and $1/2 \cdot c^{-1/2} \to \infty$ when $x \to 0$.

b) Separation of variables gives when $x > 0$ that

$$\frac{dx}{2\sqrt{x}} = dt$$

integrating we find

$$\sqrt{x} = t + c$$

hence

$$x = (t + c)^2.$$

It is a solution for $t > -c$, for $\dot{x} = 2(t+c)$ must be positive (equal to $2x^{-1/2}$ which is positive by definition). Given t_0 and x_0 the unique solution with $x(t_0) = x_0$ is

$$x(t) = (t + x_0^2 - t_0)^2$$

c) The constant function $x(t) = 0$ is obviously a solution of (*) in the region F, and satisfies $x(t_0) = 0$. Another solution with $x(t_0) = 0$ is the one given as

$$x(t) = \begin{cases} 0 & \text{for } t \leq t_0 \\ (t - t_0)^2 & \text{for } t \geq t_0 \end{cases}$$

Since $\lim_{t \to t_0^+} (t - t_0)^2 = \lim_{t \to t_0^+} 2(t - t_0) = 0$ this is continuously differentiable and it satisfies (*).

Problem 4

Consider the non-linear system

$$\begin{aligned} \dot{x} &= y - yx \\ \dot{y} &= x + yx. \end{aligned} \tag{**}$$

Show that the equilibria of the system (**) are $(0, 0)$ and $(1, -1)$, and determine the linearization of the system in each of these. Determine the type of linearizations.
Let $z(t) = (x(t), y(t))$ be a solution of the system (**). Show that $x(t) + y(t) = Ae^t$ for a suitable constant A. Show that if $z(0)$ lies on the line L given by $y + x = 0$, then $z(t)$ lies there for all t.
Let (a, b) be a point *not* lying on the line L, and let $z(t)$ be the solution with $z(0) = (a, b)$. Show that $z(t)$ is not bounded when $t \to \infty$.

a) The equilibria are the solutions of the equations

$$\begin{aligned} y - yx &= 0 \\ x + yx &= 0, \end{aligned}$$

or

$$\begin{aligned} y(1 - x) &= 0 \\ x(1 + y) &= 0. \end{aligned}$$

Hence the equilibria are $(0, 0)$ and $(1, -1)$

The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} -y & 1 - x \\ 1 + y & x \end{pmatrix}$$

In $(0,0)$ one has

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues ± 1. Hence the linearization is a saddle. In $(1,-1)$ one finds

$$J(1,-1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is an unstable node.

b) Adding the two equation in (**) gives $\dot{x} + \dot{y} = x + y$; That is $x(t) + y(t) = Ae^t$, for a constant A. If $x(0) + y(0) = 0$, it follows that $A = 0$, hence $x(t) + y(t) = 0$ for all t, and $z(t)$ lies on L for all t.

c) If $z(t)$ were bounded, one could find a constant M such that $|x(t)| < M$ and $|y(t)| < M$ for all t, but as $z(0)$ does not lie on L, one has $x(t) + y(t) = Ae^t$ with $A \neq 0$. Hence the inequality

$$|A|\, e^t = |x(t) + y(t)| \leq |x(t)| + |y(t)| \leq 2M$$

would be true for all t, which is impossible since $|A|\, e^t \to \infty$ when $t \to \infty$.

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