MAT 2500 13.10.2020

13 - 1 $\ell \in \mathbb{R}^2$ gitt

① Skal finne $L \in \mathbb{P}^2$ når $\mathbb{R}^2$ er U_2, $x_2 = 1$

② Skal finne punktet i det uendelig fjerne på L; $L \cap (x_2 = 0)$

a) $y = -\frac{x}{3}$

Ifølge eks.: $m = -\frac{1}{3}$

Punktet i det uendelig fjerne $(1 : -\frac{1}{3} : 0)$

$\frac{x_1}{x_2} = -\frac{1}{3}\frac{x_0}{x_2}$

$x_1 = -\frac{1}{3}x_0$

$L: \quad x_0 + 3x_1 = 0$

$L_\infty: \quad x_2 = 0$

$L \cap L_\infty: \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$x_0 = -3x_1 = -3t$
$x_1 = t$
$x_2 = 0$

$\downarrow$

$L \cap L_\infty = (-3 : 1 : 0) \quad (t = 1)$

$\sim (1 : -\frac{1}{3} : 0) \quad (t = -\frac{1}{3})$

b) $x = 2$

$$\frac{x_0}{x_2} = 2$$

$$x_0 = 2x_2$$

$$\left.\begin{array}{l} L: \; x_0 - 2x_2 = 0 \\ \hline L_\infty: \quad x_2 = 0 \end{array}\right\}$$

$$L \cap L_\infty : \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} \underline{1} & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \end{array}\right) \quad \begin{array}{l} x_0 = 0 \\ x_2 = 0 \\ x_1 = t \end{array}$$

$$\underline{\underline{P = L \cap L_\infty = (0:1:0)}}$$

c) $y = 4$

$$\frac{x_1}{x_2} = 4$$

$$x_1 = 4x_2$$

$$\begin{array}{l} L: \; x_1 - 4x_2 = 0 \\ \hline L_\infty \quad x_2 = 0 \end{array}$$

$$P = L \cap L_\infty : \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_0 = t \end{array} \qquad P = (1:0:0)$$

d) $\overset{m=1}{y} = x + 2$

$$\frac{x_1}{x_2} = \frac{x_0}{x_2} + 2$$

$$x_1 = x_0 + 2x_2$$

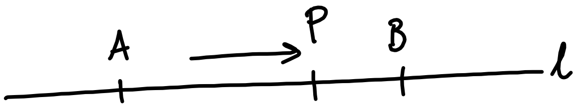
$P = (1:1:0)$

$$\begin{array}{l} L: \; x_0 - x_1 + 2x_2 = 0 \\ \hline L_\infty \quad x_2 = 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} \underline{1} & -1 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \end{array}\right)$$

$$\begin{array}{l} x_2 = 0 \\ x_1 = t \\ x_0 = x_1 = t \end{array} \qquad \underline{\underline{P = (1:1:0)}}$$

② $\mathbb{R}^2$



$$\ell = \{\underset{\text{Punkt}}{A} + \underset{\text{skalerer retning}}{t}\,\underset{\text{retning}}{(B-A)},\ t \in \mathbb{R}\}$$

$$P = A + t_0(B-A) \text{ for en } t_0 \in \mathbb{R} \setminus \overset{P \neq A,B}{\{0,1\}}$$

Skal finne $\dfrac{\overline{AP}}{\overline{PB}}$

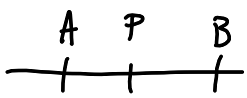
① Anta P mellom A og B : $t_0 \in (0,1)$

$$\vec{AP} = P - A = A + t_0(B-A) - A = t_0(B-A)$$

$$\overline{AP} = t_0 \|\underset{\neq 0}{B-A}\|$$

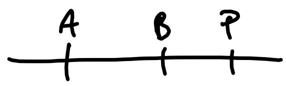
$$\vec{PB} = B - P = B - A - t_0(B-A) = (1-t_0)(B-A)$$

$$\overline{PB} = (1-t_0)\|B-A\|$$



$$\frac{\overline{AP}}{\overline{PB}} = \frac{t_0 \cancel{\|B-A\|}}{(1-t_0)\cancel{\|B-A\|}} = \frac{t_0}{1-t_0} > 0$$

② Anta A, B, P er rekkefølgen : $t_0 > 1$

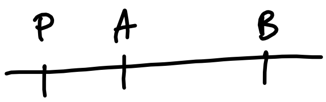


$$\vec{AP} = \underline{t_0}(B-A) \qquad \vec{PB} = \underline{(1-t_0)}(B-A)$$

$$\overline{AP} = t_0\|B-A\| \qquad \overline{PB} = (1-t_0)\|B-A\|$$

$$\frac{\overline{AP}}{\overline{PB}} = \frac{t_0}{1-t_0} < 0$$

③ $t_0 < 0$ $\qquad \dfrac{t_0}{1-t_0} < 0$



b) T invertibel lineær operator på $\mathbb{R}^2$

Skal vise $\dfrac{\overline{T(A)T(P)}}{\overline{T(P)T(B)}} = \dfrac{\overline{AP}}{\overline{PB}}$

$$\underbrace{\vec{T(A)T(P)}}_{\text{Linja gjennom } T(A) \text{ og } T(P)} = T(P) - T(A)$$

$$= T(A + t_0(B-A)) - T(A)$$

$$= \underline{T(A)} + t_0 T(B-A) \underline{- T(A)}$$

$$= t_0 T(B-A)$$

$$\overline{T(A)T(P)} = t_0 \| T(B-A) \|$$

$$\vec{T(P)T(B)} = T(B) - T(P)$$

$$= T(B) - T(A + t_0(B-A))$$

$$= \underline{T(B) - T(A)} - \underline{t_0 T(B) + t_0 T(A)} = T(B-A) - t_0 T(B-A)$$

$$= (1 - t_0) T(B-A)$$

$$\overline{T(P)T(B)} = (1-t_0)\|T(B-A)\|$$

$B - A \neq 0$, så $T(B-A) \neq 0$ siden T invertibel

$$\frac{\overline{T(A)T(P)}}{\overline{T(P)T(B)}} = \frac{t_0 \|T(B-A)\|}{(1-t_0)\|T(B-A)\|} = \frac{t_0}{1-t_0} = \frac{\overline{AP}}{\overline{PB}}$$

c) t translasjon med en vektor C i $\mathbb{R}^2$

Skal vise $\dfrac{\overline{t(A)t(P)}}{\overline{t(P)t(B)}} = \dfrac{\overline{AP}}{\overline{PB}}$

$$
\begin{aligned}
t(A)t(P) &= t(P) - t(A) \\
&= P + C - (A + C) \\
&= P - A \\
&= t_0(B - A)
\end{aligned}
$$

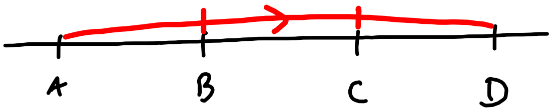
$$
\begin{aligned}
t(P)t(B) &= t(B) - t(P) \\
&= B + C - (P + C) \\
&= B - P \\
&= (1 - t_0)(B - A)
\end{aligned}
$$

og vi har

$$
\frac{\overline{t(A)t(P)}}{\overline{t(P)t(B)}} = \frac{t_0 \|B - A\|}{(1 - t_0)\|B - A\|} = \frac{t_0}{1 - t_0} = \frac{\overline{AP}}{\overline{PB}}
$$

4.3.1 Hvis A, B, C, D er kollineære, så er:

$$\overline{AD}\cdot\overline{BC} + \overline{BD}\cdot\overline{CA} + \overline{CD}\cdot\overline{AB} = 0$$



(1) $\overline{AD}\cdot\overline{BC} = (\overline{AB} + \overline{BC} + \overline{CD})\cdot\overline{BC}$

(2) $\overline{BD}\cdot\overline{CA} = (\overline{BC} + \overline{CD})\cdot(-\overline{AC})$

$= -(\overline{BC} + \overline{CD})\cdot(\overline{AB} + \overline{BC})$

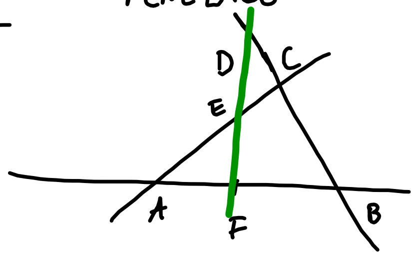
(3) $\overline{CD}\cdot\overline{AB}$

$$\begin{aligned}(1)+(2)+(3) = {} & \overline{AB}\cdot\overline{BC} + \overline{BC}^2 + \overline{CD}\cdot\overline{BC} \\ & - \overline{BC}\cdot\overline{AB} - \overline{BC}^2 - \overline{CD}\cdot\overline{AB} - \overline{CD}\cdot\overline{BC} \\ & + \overline{CD}\cdot\overline{AB}\end{aligned}$$

$$= \underline{\underline{0}}$$

4.3.2.

MENELAOS'

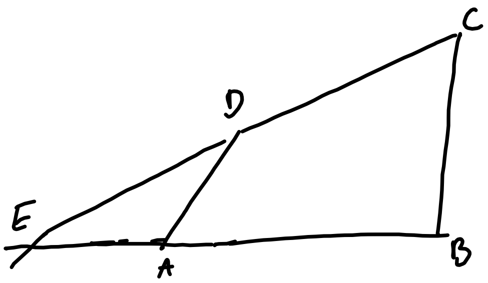


Tre Menelaos-punkter D, E, F for sidene BC, CA, AB er kollineære $\Leftrightarrow$

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1$$

GENERALISERING: For en firkant $ABCD$ vil punktene A', B', C', D', på linjene AB, BC, CD, DA, være kollineære $\Leftrightarrow$

$$\frac{\overline{AA'}}{\overline{A'B}} \cdot \frac{\overline{BB'}}{\overline{B'C}} \cdot \frac{\overline{CC'}}{\overline{C'D}} \cdot \frac{\overline{DD'}}{\overline{D'A}} = 1.$$



$\mathbb{P}^2$

ΔEBC: A' B' C' er tre Menelaos-pkt for EB BC CE
(AB) (CD)

er kollineære $\Leftrightarrow$ $$\frac{\overline{EA'}}{\overline{A'B}}\cdot\frac{\overline{BB'}}{\overline{B'C}}\cdot\frac{\overline{CC'}}{\overline{C'E}} = -1$$

ΔEAD: A' D' C' er tre Menelaospkt for sidene EA AD DE. Disse er kollineære $\Leftrightarrow$

$$\frac{\overline{EA'}}{\overline{A'A}}\cdot\frac{\overline{AD'}}{\overline{D'D}}\cdot\frac{\overline{DC'}}{\overline{C'E}} = -1$$

$$\frac{\cancel{\overline{EA'}}}{\overline{A'B}}\cdot\frac{\overline{BB'}}{\overline{B'C}}\cdot\frac{\overline{CC'}}{\cancel{\overline{C'E}}}\cdot\frac{\overline{A'A}}{\cancel{\overline{EA'}}}\cdot\frac{\overline{D'D}}{\overline{AD'}}\cdot\frac{\cancel{\overline{C'E}}}{\overline{DC'}} = (-1)(-1) = 1$$

$$\underline{\frac{\overline{AA'}}{\overline{A'B}}\cdot\frac{\overline{BB'}}{\overline{B'C}}\cdot\frac{\overline{CC'}}{\overline{C'D}}\cdot\frac{\overline{DD'}}{\overline{D'A}} = 1}$$

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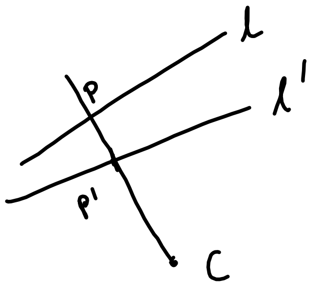
6.6.5. En PERSPEKTIVITET med senter $C \in \mathbb{P}^2$

$\alpha : l \longrightarrow l'$ mellom l og l', $l \neq l'$

$P \longmapsto PC \cap l'$

entydig, veldefinert

α bijektiv



SURJEKTIV: La $P' \in l'$

Da vil $P'C$ være en linje i $\mathbb{P}^2$

som snitter l i et punkt P.

$\alpha(P) = P'$ (siden P' er på $P'C$ og l')

$P'C = PC$

INJEKTIV:

Anta $\alpha(P) = \alpha(Q)$

$l' \cap PC = l' \cap QC = R$

P, C, R og Q, C, R er kollineære (på m)

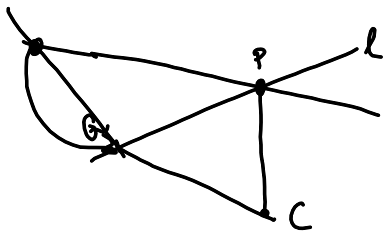
så P og Q er på den linja $m \neq l, l'$

Men P og Q er også på l.

Så $P = Q$.

α har ett fikspunkt:

EKSISTENS: $\overset{sktis}{P} = \ell \cap \ell'$ vil $\underset{\alpha(P)}{\overset{\shortparallel}{PC}} \cap \ell' = P$



ENTYDIGHET: Anta at Q er et annet f.p.

$\alpha(Q) = Q \in \ell$

$\shortparallel$

$QC \cap \ell'$

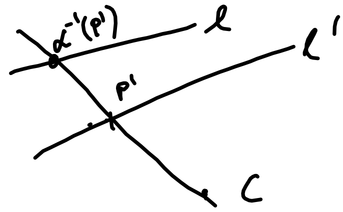
så $Q \in \ell \cap \ell'$, men da må $Q = P$.

α^{-1} er perspektivitet:

Definere $\alpha^{-1}: l' \to l$

$$P' \mapsto P'C \cap l$$

entydig siden dette er ett punkt.



$$\alpha \circ \alpha^{-1}(P') = \alpha^{-1}(P)C \cap l'$$
$$= \underline{P'}$$

Så α^{-1} er en perspektivitet