MAT 2500 27.10.2020

6.6.8 Vi skal vise at $T : \mathbb{P}^2 \to \mathbb{P}^2$ har minst ett fikspunkt

T kan gis som en invertibel 3×3-matrise

MERK: 3×3-matriser kan ha 0 som egenverdi

IKKE ha $\underline{0}$ som egenvektor (def)

T invertibel $\Rightarrow$ $\det T \neq 0$, så 0 er ikke egenverdi for T

La $\lambda \neq 0$ være egenverdi til T, da vil $1 \leq \dim E_\lambda \leq n$

og det fins egenvektor P $(\neq \underline{0})$ så $T(P) = \lambda P \sim P$

homogene koordinater

Så P er fikspunkt.

6.6.9 : $C: \underset{\parallel}{Q} = 0$ $\qquad U_2 = \{\underline{x} \in \mathbb{P}^2 : x_2 \neq 0\}$

$$a x_0^2 + b x_0 x_1 + c x_1^2 + d x_0 x_2 + e x_1 x_2 + f x_2^2$$

SETN 6.13: La $D = b^2 - 4ac$.

C snitte $\ell_\infty : x_2 = 0$ i

a) 0 pkt hvis $D < 0$ og da er $C \cap U_2$ ellipse

b) 1 pkt hvis $D = 0$ og da er $C \cap U_2$ parabel

c) 2 pkt hvis $D > 0$ og da er $C \cap U_2$ hyperbel

i) $Q = 2x_0^2 + 4x_0x_1 + 2x_1^2 - 10x_0x_2$

6.13. ① $D = 4^2 - 4 \cdot 2 \cdot 2 = 0$, så $C: Q = 0$ er en parabel

Regne eksplisitt ② $C \cap \ell_\infty$: Setter inn for ℓ_∞ i C: $x_2 = 0$ i $Q = 0$

$$2x_0^2 + 4x_0x_1 + 2x_1^2 = 0$$

Homogent polynom i 2 variabler. x_1 er ikke faktor på v.s. så setter $x_1 = 1$.

$$2x_0^2 + 4x_0 + 2 = 0$$

$$2(x_0+1)^2 = 0 \qquad x_0 = -1$$

$(1:-1:0)$

$P = (-1:1:0)$ er snittpkt mellom C og ℓ_∞.

Dette er ett punkt, altså var C en parabel i U_2.

$$x_0 = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2}$$

③ $C \cap U_2$: $\quad | : x_2^2$

$$2\left(\frac{x_0}{x_2}\right)^2 + 4\left(\frac{x_0}{x_2}\right)\left(\frac{x_1}{x_2}\right) + 2\left(\frac{x_1}{x_2}\right)^2 - 10\left(\frac{x_0}{x_2}\right)\left(\frac{x_2}{x_2}\right) = 0$$

$$2x^2 + 4xy + 2y^2 - 10x = 0$$

$$x^2 + 2xy + y^2 - 5x = 0$$

Skal bruke en isometri til å skrive denne på standard form

$$(x\ y)\underset{A}{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} - 5x = 0$$

Diagonalisering: $A = QDQ^{-1}$

Finne egenverdier til A: $\left.\begin{aligned} \lambda_1 \cdot \lambda_2 &= \det A = 0 \\ \lambda_1 + \lambda_2 &= \operatorname{tr} A = 1 + 1 = 2 \end{aligned}\right\} \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 2 \end{aligned}$

Finne Q som matrisen av ortonormale egenvektorer:

$$E_{\lambda_1}: \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \underline{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \underline{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$E_{\lambda_2}: \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Velger $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{3\pi}{4}\right) & -\sin\left(\frac{3\pi}{4}\right) \\ \sin\left(\frac{3\pi}{4}\right) & \cos\left(\frac{3\pi}{4}\right) \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = Q \cdot \begin{pmatrix} X \\ Y \end{pmatrix} \quad \rightsquigarrow \quad x = \frac{1}{\sqrt{2}}(-X - Y)$$

$$(x\ y) A \begin{pmatrix} x \\ y \end{pmatrix} - 5x = 0$$

$$\left(Q\begin{pmatrix} X \\ Y \end{pmatrix}\right)^T A \left(Q\begin{pmatrix} X \\ Y \end{pmatrix}\right) - \frac{5}{\sqrt{2}}(-X - Y) = 0$$

$$(X\ Y) \cdot \underbrace{Q^T A Q}_{\substack{\parallel \\ D \\ \parallel \\ \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}}} \begin{pmatrix} X \\ Y \end{pmatrix} + \frac{5}{\sqrt{2}}(X + Y) = 0$$

$A = QDQ^{-1}$

$D = Q^{-1}AQ$

$= Q^T A Q$

$$2Y^2 + \frac{5}{\sqrt{2}}Y = -\frac{5}{\sqrt{2}}X$$

$$\underbrace{Y^2 + \frac{5\sqrt{2}}{4}Y} = -\frac{5\sqrt{2}}{4}X$$

$$\left(Y + \frac{5\sqrt{2}}{8}\right)^2 = \left(\frac{5\sqrt{2}}{8}\right)^2 - \frac{5\sqrt{2}}{4}X$$

$$= \frac{5\sqrt{2}}{4}\left(\frac{5\sqrt{2}}{16} - X\right)$$

ii) C gitt ved $\underline{2}x_0^2 \underline{-3}x_0x_1 \underline{+}x_1^2 - 5x_0x_2 - 2x_1x_2 + 6x_2^2 = 0$

$C: \; Q = 0$

① $D = (-3)^2 - 4 \cdot 2 \cdot 1 = 1 > 0,$

så $U_2 \cap C$ er en hyperbel, $C \cap l_\infty$ er to punkter.

② $C \cap l_\infty$: $2x_0^2 - 3x_0x_1 + x_1^2 = 0$ $\quad \underline{x_1 = 1}$

$x_2 = 0$ $\quad 2x_0^2 - 3x_0 + 1 = 0$

$$x_0 = \frac{3 \pm \sqrt{9 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}$$

$\underline{x_0 = 1} \quad \wedge \quad \underline{x_0 = \frac{1}{2}}$

$P_1 = (1 : 1 : 0)$ og $P_2 = (\frac{1}{2} : 1 : 0)$

③ $\lambda = \frac{3 \pm \sqrt{10}}{2}$ ikke pent ....

iii) $3x_0x_1 - 15x_2^2 = 0$ $b = 3$ $a = c = 0$

① $D = 3^2 = 9 > 0$, hyperbel

② $C \cap l_\infty$: $P_1 = (0:1:0)$ og $P_2 = (1:0:0)$

$3x_0x_1 = 0$

③ $xy - 5 = 0$

$$\underbrace{(x\ y)\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}}_{}\begin{pmatrix} x \\ y \end{pmatrix} - 5 = 0$$

$A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

Egenverdier: $0 = \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = \lambda^2 - \frac{1}{4}$ $\underline{\lambda_1 = \frac{1}{2}}$ $\underline{\lambda_2 = -\frac{1}{2}}$

Egenvektorer:

E_{λ_1}: $\left(\begin{array}{cc|c} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

E_{λ_2}: $\left(\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$$

$D = Q^TAQ$

$\begin{pmatrix} x \\ y \end{pmatrix} = Q\begin{pmatrix} X \\ Y \end{pmatrix}$ $(X\ \ Y)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 5$

$$\frac{1}{2}X^2 - \frac{1}{2}Y^2 = 5$$

$$\underline{\frac{X^2}{10} - \frac{Y^2}{10} = 1}$$

6.6.10 • Polaritet på $\mathbb{P}^2$

$b(\underline{x},\underline{y}) = \underline{x}^t B \underline{y}$

• Polaren til P : $\{Q : b(Q,P) = 0\}$

$\uparrow$ Polen

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $P_1 = (1:0:0)$

$\boxed{\begin{array}{l} P_2 = (p_0^2 : p_1^2 : p_2^2) \\ P_3 = (p_0^3 : p_1^3 : p_2^3) \end{array}}$

P_2 og P_3 på polaren til P_1

$i=2,3$: $b(P_i, P_1) = 0 = (p_0^i \ p_1^i \ p_2^i)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$= (p_0^i \ p_1^i \ p_2^i)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$= p_0^i$

$P_2 = (0 : p_1^2 : p_2^2)$ $P_3 = (0 : p_1^3 : p_2^3)$

$\boxed{x_0 = 0}$ er Polaren til P_1 ....

Linja P_1P_2 : $0 = \begin{vmatrix} x_0 & x_1 & x_2 \\ 1 & 0 & 0 \\ 0 & p_1^2 & p_2^2 \end{vmatrix} = -p_2^2 x_1 + p_1^2 x_2$

Polaren til P_3 : $0 = b(\underline{x}, P_3) = (x_0 \ x_1 \ x_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ p_1^3 \\ p_2^3 \end{pmatrix}$

$= (x_0 \ x_1 \ x_2)\begin{pmatrix} 0 \\ p_1^3 \\ -p_2^3 \end{pmatrix}$

$= p_1^3 x_1 - p_2^3 x_2$

• $(p_1^3 + p_2^2)x_1 = (p_1^2 + p_2^3)x_2$

$p_1^3 = -p_2^2$ og $p_1^2 = -p_2^3$

Tilsvarende får vi fra P_1P_3 er lik polaren til P_2.

Homogene koordinater: kan velge $p_1^2 = 1$

$P_2 : (0 : 1 : a)$, $a \in \mathbb{R}$

$P_3 : (0 : -a : -1) \sim (0 : a : 1)$, $a \in \mathbb{R}$

• $P_1 P_2 P_3$ ikke på linje

$0 \neq \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & a & 1 \end{vmatrix} = 1 - a^2$ $a \neq \{1, -1\}$

$P_2 = P_3$

• $P_2 = (0 : 1 : a)$ og $P_3 = (0 : a : 1)$, $a \in \mathbb{R} \setminus \{1, -1\}$

Alternativ løsning:

B representerer kvadratisk form

$$(x_0 \ x_1 \ x_2)\, B \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = x_0^2 + x_1^2 - x_2^2$$

Dvs. kurven

$$C: \underbrace{x_0^2 + x_1^2 - x_2^2}_{Q} = 0$$

(for U_2 er dette vanlig sirkel)

$P = (y_0 : y_1 : y_2)$

Polaren til C mhp P

$$P_P C: \frac{\partial Q}{\partial x_0} \cdot y_0 + \frac{\partial Q}{\partial x_1} \cdot y_1 + \frac{\partial Q}{\partial x_2} \cdot y_2 = 0$$

$$2x_0 \cdot y_0 + 2x_1 \cdot y_1 + 2x_2 \cdot y_2 = 0$$

$P_1 = (1:0:0)$

$$P_{P_1} C : \underline{x_0 = 0}$$

$$\underline{P_{P_1} C} \cap C : \quad x_1^2 = x_2^2 \quad : \quad x_1 = \pm x_2$$

$(0:-1:1)$ og $(0:1:1)$