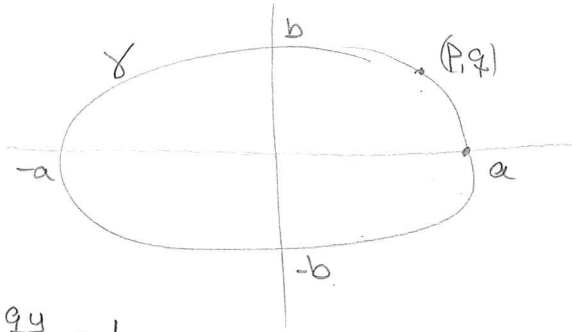


Forelesning 7. oktober

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



— Tangentlinje i (P, q) : $\frac{Px}{a^2} + \frac{qy}{b^2} = 1$

Pol $P = (x_0, y_0) \iff$ Polare $\ell(P): \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

Setning (5.4).

Et punkt P ligger på sin polare $\ell(P)$ mhp et kjeglesnitt hvis og bare hvis P ligger på kjeglesnittet og $\ell(P)$ er tangentlinja til kjeglesnittet i P .

Beris Hvis $P = (x_0, y_0) \in \ell(P)$ så er $\frac{x_0 x_0}{a^2} + \frac{y_0 y_0}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$

dvs $P \in \gamma$. Og motsatt.

Setning (5.5)

$Q \in \ell(P) \iff P \in \ell(Q)$

Beris (analytisk)

Beris for ellipse

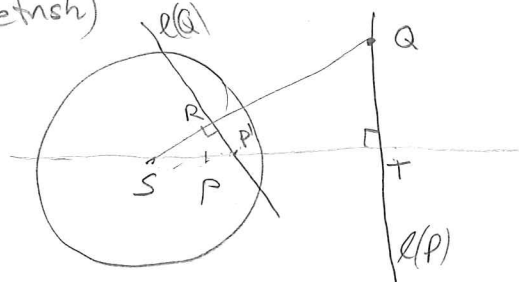
$P = (x_0, y_0) \quad \ell(P): \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

Hvis $Q = (x_1, y_1) \in \ell(P)$ så har vi

$\frac{x_0 x_1}{a^2} + \frac{y_0 y_1}{b^2} = 1$

Ved symmetri har vi $P \in \ell(Q)$

(geometrisk)



Vi har

$SR \cdot SQ = SP \cdot ST = 1$

$\Rightarrow \frac{SQ}{ST} = \frac{SP}{SR}$

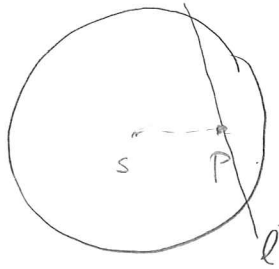
Vi har også

$\triangle SRP' \sim \triangle STQ$

$\Rightarrow \frac{SQ}{ST} = \frac{SP'}{SR}$

og det følger at $SP' = SP$ og derfor $P = P'$

Konsekvens

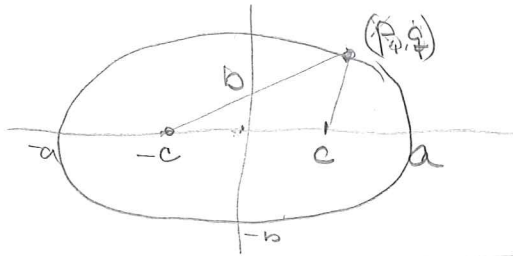


$$Q = Q(l) \Rightarrow l = l(Q)$$

Siden $P \in l(Q)$ så vil

$Q \in l(P)$.

Gjelder for alle linjer gjennom P.



$$c^2 = a^2 - b^2$$

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1$$

$$D^2 = \sqrt{q^2 + (p+c)^2} + \sqrt{q^2 + (p-c)^2}$$

$$= \sqrt{b^2 - \frac{b^2 p^2}{a^2} + p^2 + 2pc + c^2} + \sqrt{b^2 - \frac{b^2 p^2}{a^2} + p^2 - 2pc + c^2}$$

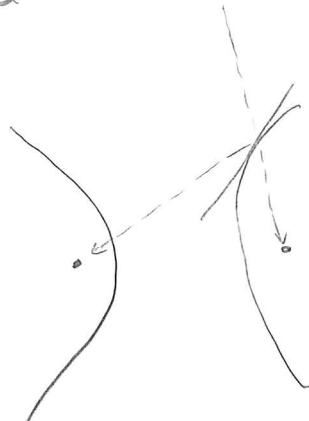
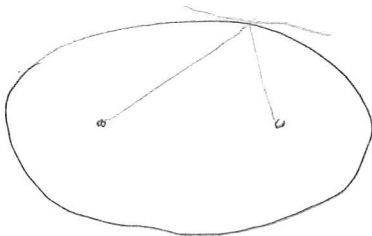
$$= \sqrt{a^2 + \frac{p^2}{a^2} c^2 + 2pc} + \sqrt{a^2 + \frac{p^2}{a^2} c^2 - 2pc}$$

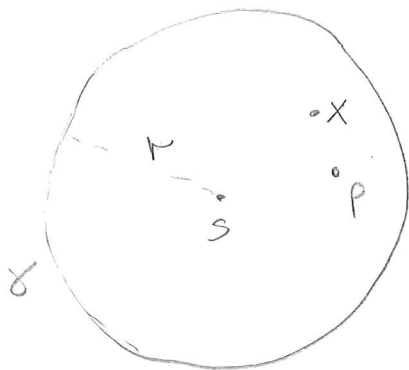
$$D^2 = a^2 + \frac{p^2}{a^2} c^2 + 2pc + 2 \underbrace{\sqrt{\left(a^2 + \frac{p^2}{a^2} c^2\right)^2 - (2pc)^2}}_{(*)} + a^2 + \frac{p^2}{a^2} c^2 - 2pc$$

$$(*) \quad a^4 + 2p^2 c^2 + \frac{p^4}{a^4} c^4 - 4p^2 c^2 = \left(a^2 - \frac{p^2}{a^2} c^2\right)^2$$

$$D^2 = 2a^2 + 2 \frac{p^2}{a^2} c^2 + 2a^2 - 2 \frac{p^2}{a^2} c^2 = 4a^2 \Rightarrow \underline{\underline{D = 2a}}$$

Spåling:





Geometrisk sted:

Alle punkter med samme avstand til P som til Y.

Ellipse med brennpunkter S og P, summen av avstandene er r.

$$\|SX\| + \|XP\| = r - d(X,Y) + \|XP\| = r$$

$$\Rightarrow d(X,Y) = \|XP\|$$

Kvadratiske likninger

$$5x^2 + 6xy + 5y^2 - 10x - 6y + 1 = 0$$

1) Fjerne kryss ledd

$$(x \ y) \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5x^2 + 6xy + 5y^2$$

$$\text{Kar. pol. } \lambda^2 - 10\lambda + 16 = 0 \Rightarrow \lambda = \frac{10 \pm \sqrt{100 - 64}}{2} = \begin{cases} 8 \\ 2 \end{cases}$$

$$\text{Egenvektorer. } \lambda_1 = 8 : \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2 : \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{Setter } u = \frac{1}{\sqrt{2}}(x+y) \quad v = \frac{1}{\sqrt{2}}(x-y), \text{ som gir } x = \frac{1}{\sqrt{2}}(u+v) \quad y = \frac{1}{\sqrt{2}}(u-v)$$

$$5\left(\frac{u+v}{2}\right)^2 + 6\left(\frac{u+v}{2}\right)\left(\frac{u-v}{2}\right) + 5\left(\frac{u-v}{2}\right)^2 - 10\frac{u+v}{\sqrt{2}} - 6\frac{u-v}{\sqrt{2}} + 1 = 0$$

$$\frac{5}{2}u^2 + 5uv + \frac{5}{2}v^2 + 3u^2 - 3v^2 + \frac{5u^2}{2} - 5uv + \frac{5v^2}{2} - \frac{10}{\sqrt{2}}u - \frac{10}{\sqrt{2}}v - \frac{6}{\sqrt{2}}u + \frac{6}{\sqrt{2}}v + 1 = 0$$

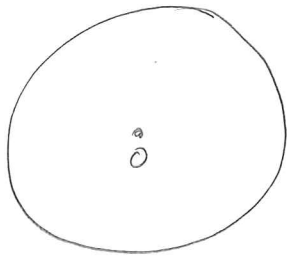
$$8u^2 + 2v^2 - \frac{16}{\sqrt{2}}u - \frac{4}{\sqrt{2}}v + 1 = 0$$

2) Lage fullstendig kvadrat

$$8\left(u^2 - \frac{2}{\sqrt{2}}u + \frac{1}{2}\right) + 2\left(v^2 - \frac{2}{\sqrt{2}}v + \frac{1}{2}\right) - 4 - 1 + 1 = 0$$

$$8\left(\underbrace{u - \frac{1}{\sqrt{2}}}_z\right)^2 + 2\left(\underbrace{v - \frac{1}{\sqrt{2}}}_w\right)^2 = 4$$

$$8z^2 + 2w^2 = 4 \quad \text{eller} \quad \frac{z^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{w^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1.$$



Enhetscirkel i $\mathbb{C}$
 $|z| = 1$

$$w \in \mathbb{C}$$

$$w = x_0 + iy_0$$

$$x_0 = \frac{1}{2}(w + \bar{w}) \quad y_0 = \frac{1}{2i}(w - \bar{w})$$

$$\text{Polare: } xx_0 + yy_0 = 1$$

$$x + iy = z$$



$\Downarrow$

$$\frac{1}{2}(z + \bar{z}) \frac{1}{2}(w + \bar{w})$$

$$x = \frac{1}{2}(z + \bar{z})$$

$$y = \frac{1}{2i}(z - \bar{z})$$

$$+ \frac{1}{2}(z - \bar{z}) \frac{1}{2i}(w - \bar{w}) = 1$$

$$zw + \cancel{z\bar{w}} + \cancel{\bar{z}w} + \bar{z}\bar{w}$$

$$+ zw - \cancel{z\bar{w}} - \cancel{\bar{z}w} + \bar{z}\bar{w} = 4$$

$$\frac{1}{2}(zw + \bar{z}\bar{w}) = 1$$

$$\text{Polaren til } w \text{ er } \{z \in \mathbb{C} \mid \operatorname{Re}(zw) = 1\}$$

$$\text{Eksempel: } w \in \mathbb{R} \Rightarrow \bar{w} = w$$

$$\Rightarrow \frac{1}{2}(zw + \bar{z}w) = \frac{1}{2}(z + \bar{z})w = 1$$

$$\Rightarrow \operatorname{Re}(z) = \frac{1}{w}$$