

13. oktober 2021

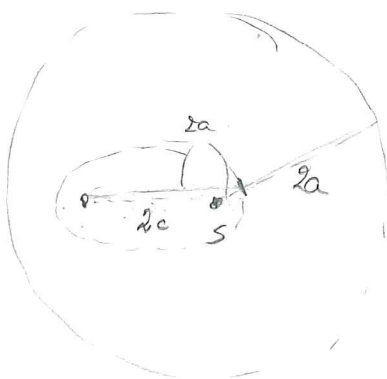
Setning 5.9.

Det geometriske stedet for de punktene som har samme avstand til en sirkel som til et gitt punkt inne i sirkelen danner en ellipse.

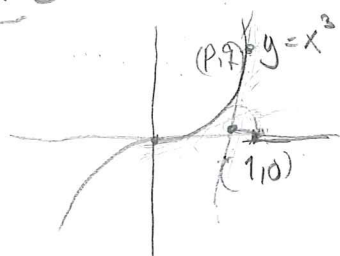
Setning 5.10



den ene delen
er en hyperbel.



Pedalkurve



Tangent $y' = 3x^2$

$$y - q = 3p^2(x - p) \quad \text{st. full } 3p^2$$

$$\text{Fotpunkt } (x_0, y_0) \quad y_0 - q = 3p^2(x_0 - p) \Rightarrow y_0 - q = 3p^2x_0 - 3p^3$$

Linje fra (1, 0) til fotpunkt:

$$y = \frac{y_0}{x_0 - 1}(x - 1)$$

Dette gir kurve

$$\text{Normalitet} \quad \frac{y_0}{x_0 - 1} \cdot 3p^2 = -1$$

$$y_0 = 3 \frac{1 - x_0}{3y_0} x_0 - 2 \left(\frac{1 - x_0}{3y_0} \right)^{3/2}$$

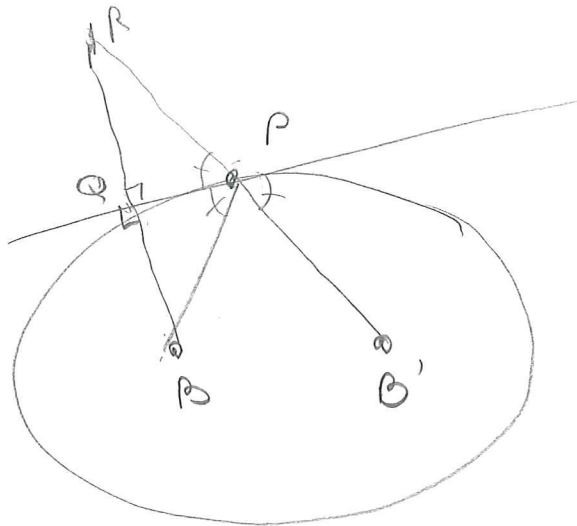
$$3p^2 y_0 = 1 - x_0 \quad -p = \sqrt{\frac{1 - x_0}{3y_0}}$$

$$y_0 = 3p^2 x_0 - 2p^3$$

$$\left(3 \frac{1 - x_0}{3y_0} x_0 - y_0 \right)^2 = 4 \left(\frac{1 - x_0}{3y_0} \right)^3$$

$$(1 - x_0)x_0 y_0 - y_0^3 = 4 \frac{1}{27} (1 - x_0)^3$$

Pedalcurve til ellipse mhp brennpunkt



$$\triangle PQB \sim \triangle PQR$$

$$\Rightarrow PB = PR$$

$$\Rightarrow B'R = 2a$$

$$\triangle OQB \sim \triangle B'RB \quad \text{fordi } \angle B \text{ felles og}$$

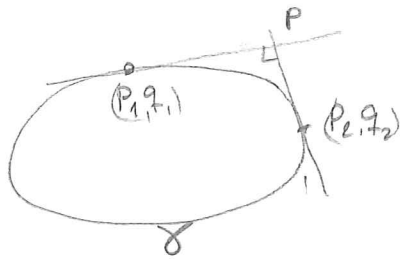
$$\frac{OB}{BB'} = \frac{1}{2} = \frac{QB}{BR}$$

$$\Rightarrow OQ = \frac{1}{2} B'R = \frac{1}{2} 2a = a$$

Storsirkelen til ellipsen



Styresittel



Tangentene til γ fra P står normalt på hverandre.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\text{Tangentlinje } y - y_i = -\frac{b^2 p_i}{a^2 y_i} (x - p_i)$$

$$b^2 p^2 + a^2 q^2 = a^2 b^2 \quad \text{To normale tangenter.}$$

$$-\frac{b^2 p_1}{a^2 q_1}, -\frac{b^2 p_2}{a^2 q_2} = -1. \Leftrightarrow \frac{p_1 p_2}{a^4} + \frac{q_1 q_2}{b^4} = 0$$

$$\frac{p_2}{q_1} = -\frac{q_2}{p_1} \frac{a^4}{b^4}$$

Skjærings mellom

$$y - q_i = -\frac{b^2}{a^2} \frac{p_i}{q_i} (x - p_i) \quad i = 1, 2$$

$$\Rightarrow y = \frac{b^2}{q_i a^2} (-p_i x + a^2)$$

$$\Rightarrow x = \frac{a^2(q_1 - q_2)}{q_1 p_2 - q_2 p_1} \quad y = \frac{b^2(p_1 - p_2)}{p_1 q_2 - p_2 q_1}$$

Som gir

$$x^2 - \frac{b^2}{a^2} y^2 = b^2 - \frac{b^4 (p_1 - p_2)^2}{(p_1 q_2 - p_2 q_1)^2} = \frac{b^2}{(p_1 q_2 - p_2 q_1)^2} ((p_1 q_2 - p_2 q_1)^2 - b^2 (p_1 - p_2)^2)$$

$$\begin{aligned} (p_1 q_2 - p_2 q_1)^2 - b^2 (p_1 - p_2)^2 &= p_1^2 q_2^2 - 2 p_1 p_2 q_1 q_2 + p_2^2 q_1^2 - b^2 p_1^2 + 2 b^2 p_1 p_2 - b^2 p_2^2 \\ &= p_1^2 \left(\frac{a^2 q_2^2}{a^2} - \frac{a^2 b^2}{a^2} \right) + p_2^2 \left(\frac{a^2 q_1^2}{a^2} - \frac{b^2 a^2}{a^2} \right) + 2 p_1 p_2 (b^2 - q_1 q_2) \\ &= -p_1^2 \frac{b^2 p_2^2}{a^2} - p_2^2 \frac{b^2 p_1^2}{a^2} + 2 p_1 p_2 b^2 - 2 p_1 p_2 q_1 q_2 \\ &= 2 p_1 p_2 \left(-\frac{b^2}{a^2} p_1 p_2 + b^2 - q_1 q_2 \right) \\ &= -\frac{2 p_1 p_2}{a^2} (b^2 p_1 p_2 + a^2 q_1 q_2 - a^2 b^2) \end{aligned}$$

$$\text{dvs. } b^2 - y^2 = \frac{-2 b^2 p_1 p_2}{a^2 (p_1 q_2 - p_2 q_1)^2} (b^2 p_1 p_2 + a^2 q_1 q_2 - a^2 b^2)$$

$$a^2 - x^2 = \frac{-2 a^2 q_1 q_2}{b^2 (p_1 q_2 - p_2 q_1)^2} (a^2 q_1 q_2 + b^2 p_1 p_2 - a^2 b^2)$$

Men vi har

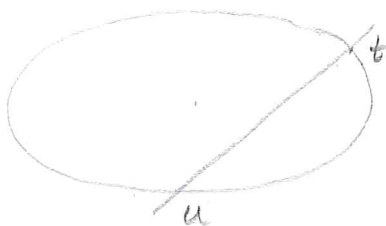
$$\begin{aligned} \frac{b^2 p_1 p_2}{a^2} + \frac{a^2 q_1 q_2}{b^2} &= \frac{a^2 b^2 p_1 p_2}{a^4} + \frac{a^2 q_1 q_2}{b^2} \\ &= \frac{-a^2 b^2 q_1 q_2}{b^4} + \frac{a^2 q_1 q_2}{b^2} = 0 \\ \Rightarrow b^2 y^2 + a^2 - x^2 &= 0 \quad \text{eller} \quad \underline{x^2 + y^2 = a^2 + b^2} \\ &\quad \text{styresirkelen} \end{aligned}$$

Setning

Det geometriske stedet for midtpunktet til parallelle korder i en ellipse er en diameter.

Bevis

Parametrisering $x = a \cos t$ $y = b \sin t$.



Midtpunkt

$$\begin{aligned} x &= \frac{1}{2} (a \cos t + a \cos u) = a \cos \frac{t+u}{2} \cos \frac{t-u}{2} \\ y &= b \sin \frac{t+u}{2} \cos \frac{t-u}{2} \end{aligned}$$

$$t = \frac{t+u}{2} + \frac{t-u}{2}$$

$$u = \frac{t+u}{2} - \frac{t-u}{2}$$

Stigningstallet :

$$m = \frac{b \sin t - b \sin u}{a \cos t - a \cos u} = \frac{2b \cos \frac{t+u}{2} \sin \frac{t-u}{2}}{-2a \sin \frac{t+u}{2} \sin \frac{t-u}{2}}$$

$$\cos(\alpha + \beta) =$$

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) =$$

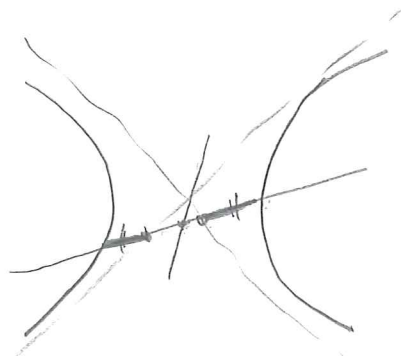
$$\sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= -\frac{b}{a} \frac{\cos \frac{t+u}{2}}{\sin \frac{t+u}{2}} = -\frac{b}{a} \frac{x}{\frac{y}{b \cos \frac{t-u}{2}}} = -\frac{b^2 x}{a^2 y}$$

$$\Rightarrow b^2 x + a^2 m y = 0$$

Denne linje går gjennom sentrum.

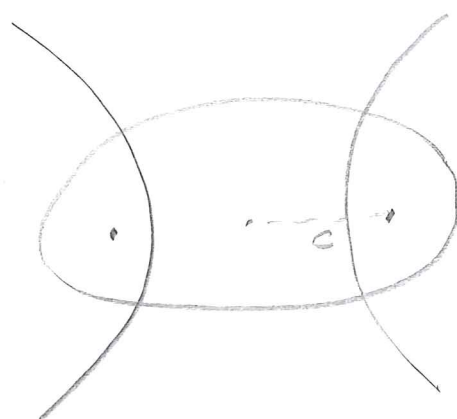
Setning



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Asymptotene : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ grense for hyperbler $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \lambda$
 $\lambda \rightarrow 0$.

Ellipse og hyperbel med felles brennpunkter



$$a_1^2 - a_1^2 - b_1^2 + b_1^2 = 0$$

$$\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$$

$$c^2 = a_1^2 + b_1^2$$

$$\frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = 1$$

$$= a_2^2 - b_2^2$$

$$h = a_2^2 - a_1^2 = b_2^2 + b_1^2$$

Hyperbel/Ellipse $\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1$

Tangent linje gjennom (p, q) :

$$\frac{px}{a^2 - \lambda} + \frac{qy}{b^2 - \lambda} = 1$$

To linjer står normalt på hverandre dersom (p, q) på begge hyskesnittene

$$\frac{p}{a^2 - \lambda_1} \cdot \frac{p}{a^2 - \lambda_2} + \frac{q}{b^2 - \lambda_1} \cdot \frac{q}{b^2 - \lambda_2} = 0$$

Men vi har

$$\frac{p^2}{a^2 - \lambda_1} + \frac{q^2}{b^2 - \lambda_1} = 1, \quad \frac{p^2}{a^2 - \lambda_2} + \frac{q^2}{b^2 - \lambda_2} = 1$$

Subtraksjon gir

$$\frac{p^2}{a^2 - \lambda_1} + \frac{q^2}{b^2 - \lambda_1} - \frac{p^2}{a^2 - \lambda_2} - \frac{q^2}{b^2 - \lambda_2} = \frac{p^2(\lambda_1 - \lambda_2)}{(a^2 - \lambda_1)(a^2 - \lambda_2)} + \frac{q^2(\lambda_1 - \lambda_2)}{(b^2 - \lambda_1)(b^2 - \lambda_2)} = 0$$

$$= (\lambda_1 - \lambda_2) \left[\frac{p^2}{(a^2 - \lambda_1)(a^2 - \lambda_2)} + \frac{q^2}{(b^2 - \lambda_1)(b^2 - \lambda_2)} \right] = 0$$

dvs. dersom $\lambda_1 \neq \lambda_2$ så vil hyllesnittene stå normalt på hverandre i styngspunkter

