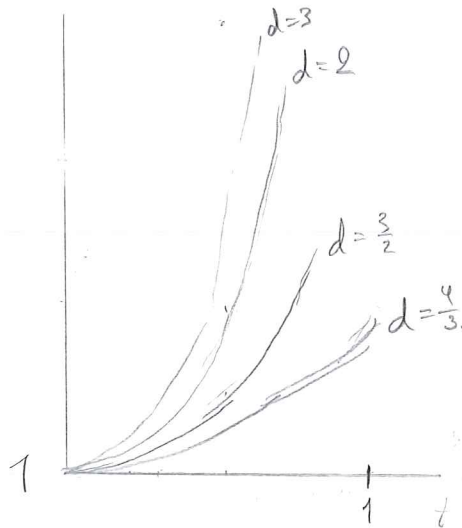


Forelesning 28. oktober



Kryssforhold: $(AB, CD) = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}}$



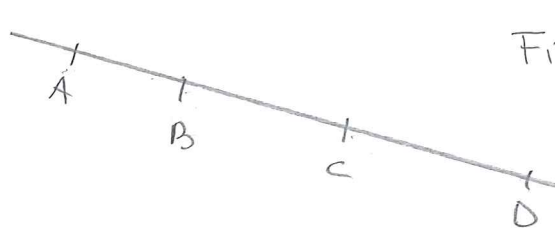
$$\begin{array}{ll} A=0 & b=t \\ C=1 & d=d \end{array}$$

$$\begin{array}{llll} A=(0,0,1) & B=(t,0,1) & C=(1,0,1) & D=(d,0,1) \end{array} \quad M = \begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix}$$

$$\begin{array}{l} \downarrow \\ (m_{02}, m_{12}, m_{22}) \end{array} \quad \begin{array}{l} \downarrow \\ (m_{02}+t m_{00}, m_{12}+t m_{10}, m_{22}+t m_{20}) \end{array} \quad \begin{array}{l} \downarrow \\ (m_{02}+d m_{00}, m_{12}+d m_{10}, m_{22}+d m_{20}) \end{array}$$

$$(AB, CD) = \frac{1}{d} \frac{d-t}{1-t} = \frac{1-\frac{t}{d}}{1-t}$$

$$\begin{aligned} (T(A)T(B), T(C)T(D)) &= \sqrt{m_{00}^2 + m_{10}^2 + m_{20}^2} \cdot \frac{1}{d \sqrt{m_{00}^2 + m_{10}^2 + m_{20}^2}} \\ &\quad \cdot (d-t) \sqrt{m_{00}^2 + m_{10}^2 + m_{20}^2} \cdot \frac{1}{(1-t) \sqrt{m_{00}^2 + m_{10}^2 + m_{20}^2}} \\ &= \frac{1-\frac{t}{d}}{1-t} \quad \text{invariant under PT.} \end{aligned}$$



Fire punkter på linje i $\mathbb{P}^2$

$$C = [pV_A + qV_B] \quad p+q=1$$

$$D = [rV_A + sV_B] \quad r+s=1$$

$$V_B, V_A \in \mathbb{R}^3 \setminus \{0\}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$B, A \in \mathbb{P}^2$$

$$(A, B; C, D) = \frac{AC}{AD} \cdot \frac{BD}{BC} = \frac{q \cdot r}{s \cdot p}$$

$$AC = \|(p-1)V_A + qV_B\|$$

$$= q \|V_B - V_A\|$$

$$AD = s \|V_B - V_A\|$$

$$BD = -r \|V_B - V_A\|$$

$$BC = -p \|V_B - V_A\|$$

Eksempel

1) $A = (-\frac{3}{2}, -2)$ $B = (3, 1)$ $C = (0, -1)$ $D = (4, \frac{5}{3})$

ligger på linja $y = \frac{2}{3}x - 1$

$$\overline{AC} = \frac{1}{2}\sqrt{13} \quad \overline{AD} = \frac{11}{6}\sqrt{13} \quad \Rightarrow (A, B; C, D) = \frac{\frac{1}{2}\sqrt{13} \cdot \frac{1}{3}\sqrt{13}}{\frac{11}{6}\sqrt{13} \cdot (-\sqrt{13})} = -\frac{1}{11}$$

$$\overline{BC} = -\sqrt{13} \quad \overline{BD} = \frac{1}{3}\sqrt{13}$$

2) $C = \frac{2}{3}A + \frac{1}{3}B$ $D = -\frac{2}{9}A + \frac{11}{9}B \Rightarrow (A, B; C, D) = \frac{\frac{1}{3} \cdot (-\frac{2}{9})}{\frac{11}{9} \cdot \frac{2}{3}} = -\frac{1}{11}$

3) $A = (3:4:-2)$ $B = (3:1:1)$ $C = (0:-1:1)$ $D = (12:5:3)$

ligger på linja $2x_0 - 3x_1 - 3x_2 = 0$

$$\begin{pmatrix} 3 & 3 \\ 4 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ -1 & 5 \\ 1 & 3 \end{pmatrix}$$

Observer at

$$\begin{pmatrix} -\frac{1}{9} & \frac{1}{3} & 0 \\ \frac{2}{9} & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 4 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

som betyr

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} -\frac{1}{9} & \frac{1}{3} & 0 \\ \frac{2}{9} & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 12 \\ -1 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{11}{3} \end{pmatrix} \Rightarrow \frac{qr}{sp} = -\frac{1}{11}$$

Sætning om kryssforhold

- 1) La $A, B, C, X, Y \in \mathbb{P}^2$ være punkter på en linje og antag

$$(A, B; C, X) = (A, B; C, Y)$$

Da er $X=Y$.

- 2) Læ. $\lambda \in \mathbb{R}$ u. $\lambda \neq 0$. Da finnes punkt $D \in \mathbb{P}^2$ på samme linje slik at

$$(A, B; C, D) = \lambda.$$

Bevis.

- 1) V. har $C = [p v_A + q v_B] \quad p+q=1$
 $X = [r_1 v_A + s_1 v_B] \quad r_1+s_1=1$
 $Y = [r_2 v_A + s_2 v_B] \quad r_2+s_2=1$

$$X = [r_1 v_A + s_1 v_B] \quad r_1 + s_1 = 1$$

$$V = [\Gamma_2 V_A + S_2 V_B] \quad \Gamma_2 + S_2 = 1$$

Del cri

$$\frac{q_{r_1}}{s_1 p} = \frac{q_{r_2}}{s_2 p} \Leftrightarrow r_1 s_2 = r_2 s_1$$

setter mm for s_1 og s_2 : $t_1(1-t_2) = t_2(1-t_1)$

$$\begin{array}{ccc} \text{11} & \text{11} & \\ r_1 - r_1 r_2 & r_2 - r_1 r_2 & \Rightarrow r_1 = r_2 \\ & & \Rightarrow X = Y \end{array}$$

- 2) p og q er gitt, dessuten $\frac{q+1}{sp} = 1$

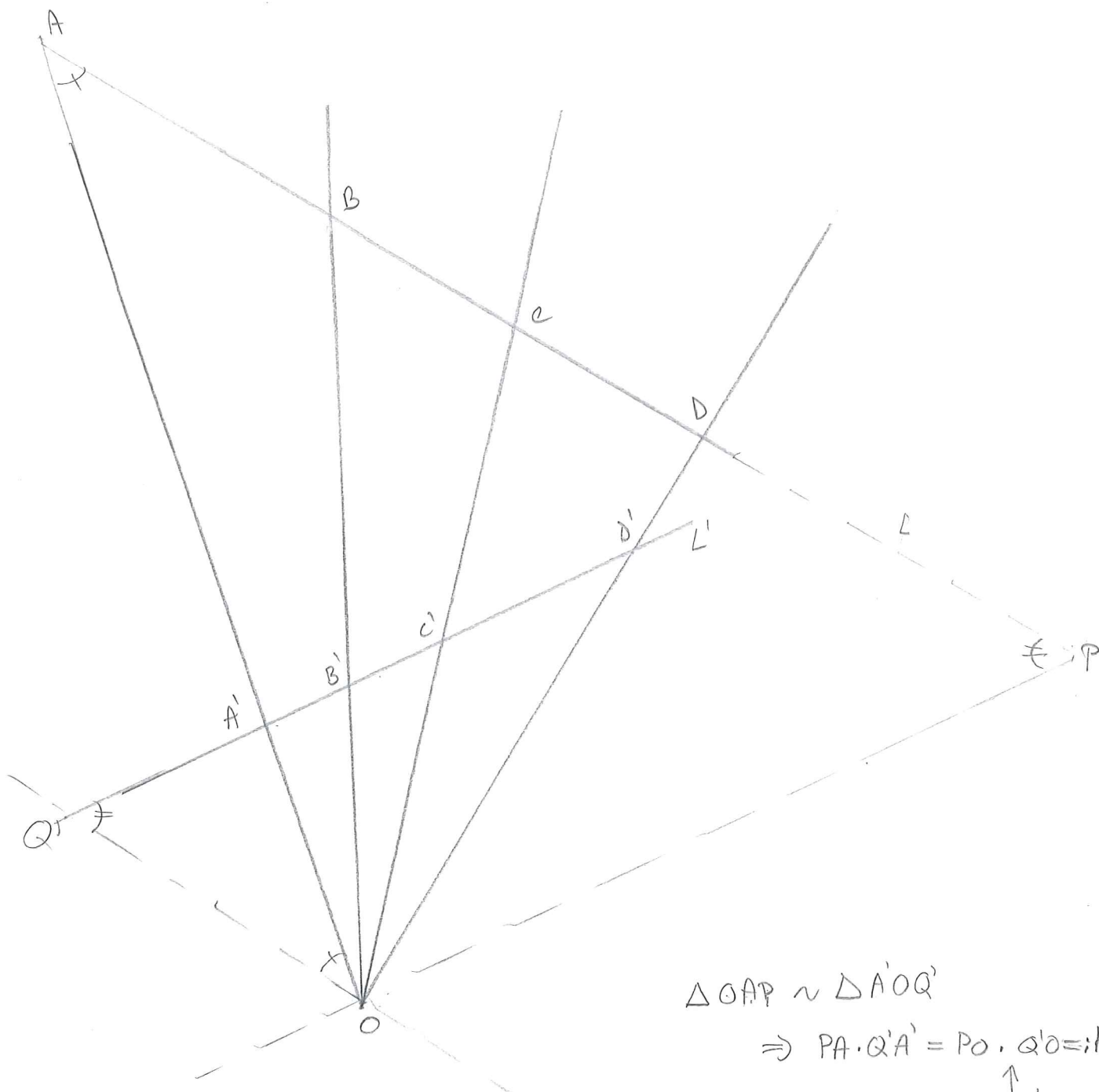
$$\text{dvs } s = \frac{q}{p_h} \cdot t, \text{ eller } s + v = \left(\frac{q}{p_h} + 1\right) t$$

Self $r = \frac{1}{\frac{q}{px} + 1} = \frac{px}{q+px}$ or $s = \frac{q}{q+px}$ or $D = [rV_A + sV_B]$

$$\Rightarrow (A, B; C, D) = \frac{q' \frac{px}{q+px}}{\frac{q}{q+px} \cdot p} = \underline{\underline{\lambda}}$$

Sætning

Krydsforhold er bevaret under perspektiv



$$\triangle OAP \sim \triangle A'OQ'$$

$$\Rightarrow PA \cdot Q'A' = PO \cdot Q'O =: k$$

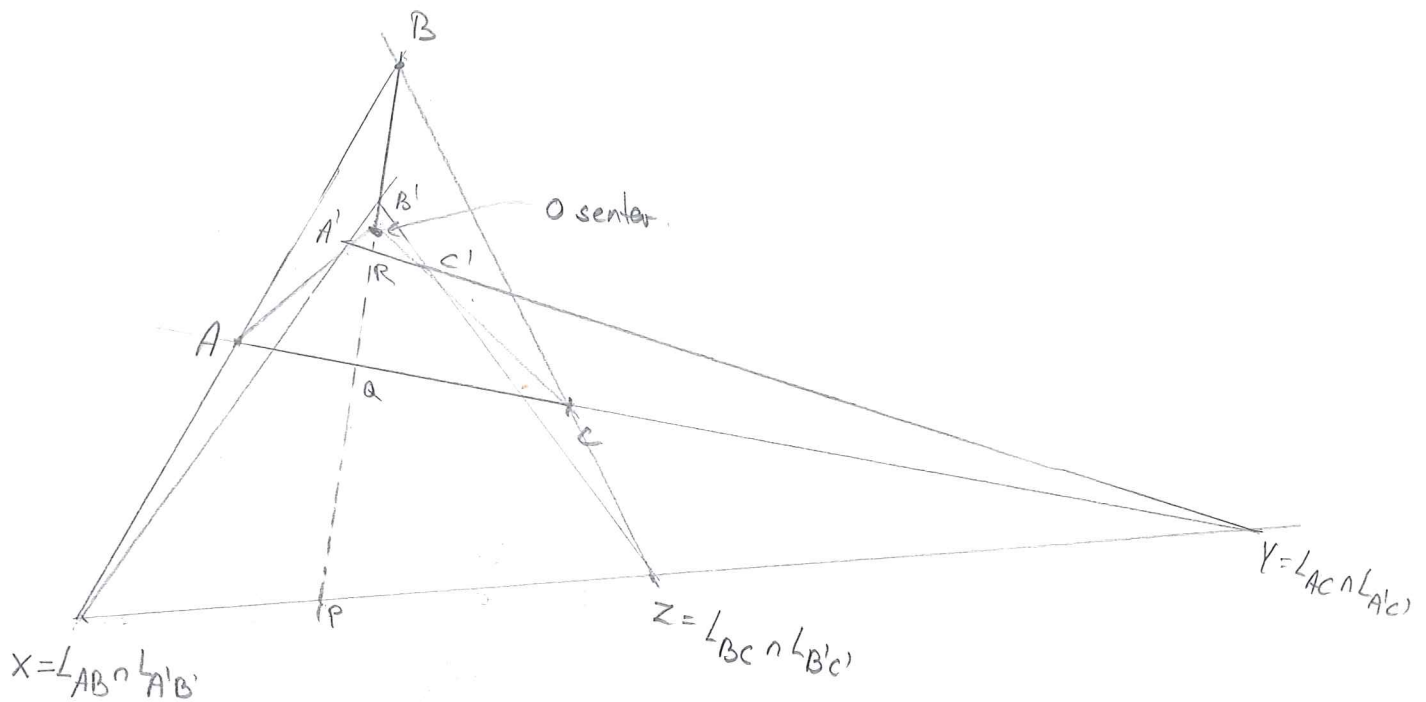
↑
uafhængig
af A, A'

$$k = PA \cdot Q'A' = PB \cdot Q'B' , \quad \begin{aligned} AB &= PA - PB \\ A'B' &= Q'A' - Q'B' \end{aligned}$$

$$\Rightarrow A'B' = \frac{k}{PA} - \frac{k}{PB} = \frac{PB \cdot k - PA \cdot k}{PA \cdot PB} = \frac{-k}{PA \cdot PB} \cdot AB$$

$$\Rightarrow \frac{A'C'}{A'D'} \cdot \frac{B'D'}{B'C'} = \frac{-k}{PA \cdot PC} \cdot \frac{PA \cdot PD}{-k} \cdot \frac{-k}{PB \cdot PD} \cdot \frac{PB \cdot PC}{-k} \cdot \frac{AC}{AD} \cdot \frac{BD}{BC} = \frac{AC}{AD} \cdot \frac{BD}{BC}$$

Desargues setning



Vi skal vise at Z ligger på L_{xy}

Definerer to punkter $S = L_{BC} \cap L_{xy}$

$S' = L_{B'C'} \cap L_{xy}$

$$(X, Y; P, S) \stackrel{B}{=} (A, Y; Q, C) \stackrel{O}{=} (A', Y; R, C') \stackrel{B'}{=} (X, Y; P, S')$$

$$\Rightarrow \underline{\underline{S = S'}}$$