

Keglesnitt i $\mathbb{P}^2$

I det affine planet:

$$Q_2 = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

1) Diagonalisere $\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \Rightarrow$ kan anta $b=0$ 2) Flytte sentrum $\Rightarrow d=e=0$

3) Skalere variable

$$x^2 + y^2 = 1 \quad x^2 - y^2 = 1 \quad y = x^2$$

Kvadratisk form:

$$\swarrow \text{Diskriminant} \quad D = b^2 - 4ac$$

$$Q = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_2^2$$

Kvadratisk kurve i $\mathbb{P}^2$: $Q=0$ for valg av a, b, c, d, e, f

Merk: Degenerert: $Q = L_1 \cdot L_2$ (L_1, L_2 lineære) $\begin{cases} x^2 - y^2 = 0 \\ x^2 + y^2 = 0 \\ x^2 = 0 \end{cases}$
 $Q=0$ gir $\emptyset$, ett punkt eller linje

Snitt med U_2 ($x_2 \neq 0$) gir med $x = \frac{x_0}{x_2}$, $y = \frac{x_1}{x_2}$ Q_2 tilbake.

Setning:

 $C \subseteq \mathbb{P}^2$
 ikke-degenerert
 kvadratisk kurve

$$l_\infty: x_2 = 0$$

1) $D < 0 \Rightarrow C \cap l_\infty = \emptyset$ C : ellipse2) $D = 0 \Rightarrow C \cap l_\infty = \{\text{pt}\}$ C : parabel3) $D > 0 \Rightarrow C \cap l_\infty = \{p, q\}$ C : hyperbel

Bevis

$$C \cap l_\infty = \{(x_0: x_1: 0) \mid ax_0^2 + bx_0x_1 + cx_1^2 = 0\}$$

Kan anta $x_1 \neq 0$, dvs $x_1 = 1$

$$= \{(x: 1: 0) \mid ax^2 + bx + c = 0\} \quad D: \text{vanlig diskriminant}$$

Hva slags type kjeslesnitt:

Bestemt av fortegnet til egenverdiene til $\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$

$$\det \begin{pmatrix} a-\lambda & \frac{b}{2} \\ \frac{b}{2} & c-\lambda \end{pmatrix} = \lambda^2 - (a+c)\lambda + ac - \frac{b^2}{4} = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

$$\Rightarrow \lambda_1, \lambda_2 = \frac{a+c}{2} \pm \sqrt{\frac{(a+c)^2 - 4ac + b^2}{4}}$$

$$= \frac{a+c}{2} \pm \sqrt{\frac{(a-c)^2 + b^2}{4}}$$

$$\begin{aligned} -4\lambda_1\lambda_2 &= -4\left(\left(\frac{a+c}{2}\right)^2 - \frac{(a+c)^2 - 4ac + b^2}{4}\right) \\ &= -4 \cdot \frac{1}{4} (a^2 + 2ac + c^2 - a^2 - 2ac - c^2 + 4ac - b^2) \\ &= b^2 - 4ac = D \end{aligned}$$

$$\Rightarrow D > 0 \Leftrightarrow \lambda_1, \lambda_2 \text{ har ulikt tegn}$$

$$D = 0 \Leftrightarrow \lambda_1 = 0 \text{ eller } \lambda_2 = 0$$

$$D < 0 \Leftrightarrow \lambda_1, \lambda_2 \text{ har samme fortegn.}$$

Vi kan skrive

$$Q = (x_0 \ x_1 \ x_2) \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \quad (x_0 \ x_1 \ x_2) \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

symmetrisk $\Rightarrow$ kan ortogonalt (ortonormalt) diagonaliseres

$\Rightarrow$ Ved projektiv transform: $Q = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2$

Muligheter:

$$\lambda_0 = \lambda_1 = 0 \Rightarrow Q = \lambda_2 x_2^2 \text{ degenerert.}$$

$$\lambda_0 = 0 \ \lambda_1, \lambda_2 > 0 \Rightarrow x_1^2 + x_2^2 = 0 \text{ ett punkt, degenerert, } (1; 0; 0)$$

$$\lambda_0 = 0 \ \lambda_1, \lambda_2 < 0 \Rightarrow x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) \text{ degenerert.}$$

$$\lambda_0, \lambda_1, \lambda_2 > 0 \Rightarrow x_1^2 + x_2^2 + x_3^2 = 0 \text{ Ingen løsning}$$

$$\lambda_0, \lambda_1 > 0 \ \lambda_2 < 0 \Rightarrow x_0^2 + x_1^2 - x_2^2 = 0 \text{ ikke-degenerert.}$$

Hyperbler, Parabler og ellipser er projektivt det samme.

5 pkt i $\mathbb{P}^2$ i generell posisjon bestemmer en kvadratisk kurve

Kan anta punktene er

$$(1:0:0), (0:1:0), (0:0:1), (1:1:1), (x_0:x_1:x_2)$$

$$Q(1,0,0) = a = 0 \quad Q(0,1,0) = c = 0 \quad Q(0,0,1) = f = 0$$

$$Q(1,1,1) = a + b + c + d + e + f = 0$$

$$\text{dvs. } a = c = f = b + d + e = 0$$

$$\begin{aligned} x_0(x_2 - x_1) &= 0 \\ x_1(x_2 - x_0) &= 0 \\ x_2(x_1 - x_0) &= 0 \end{aligned}$$

maa.

$$\begin{pmatrix} 1 & 1 & 1 \\ x_0 x_1 & x_0 x_2 & x_1 x_2 \end{pmatrix} \begin{pmatrix} b \\ d \\ e \end{pmatrix} = 0$$

og minst én minor $\neq 0$
 $x_0(x_2 - x_1) = 0$

Har entydig projektiv løsning dersom ikke $x_0 x_1 = x_0 x_2 = x_1 x_2 = 0$

$$\text{dvs. } x_0 = x_1 = 0. \text{ Men det betyr } (x_0:x_1:x_2) = (0:0:1)$$

Som er umulig (kun 4 punkter).

Hvorfor kjeglesnitt?

$$x^2 + y^2 = z^2$$

Kjegle.



plan gjennom linje

$$z = 0 \quad y = 1$$

$$z = y - 1 = 0$$

$$\frac{y}{x} = \tan \alpha$$

$$x = u, y = v \cos \alpha + 1, z = v \sin \alpha$$

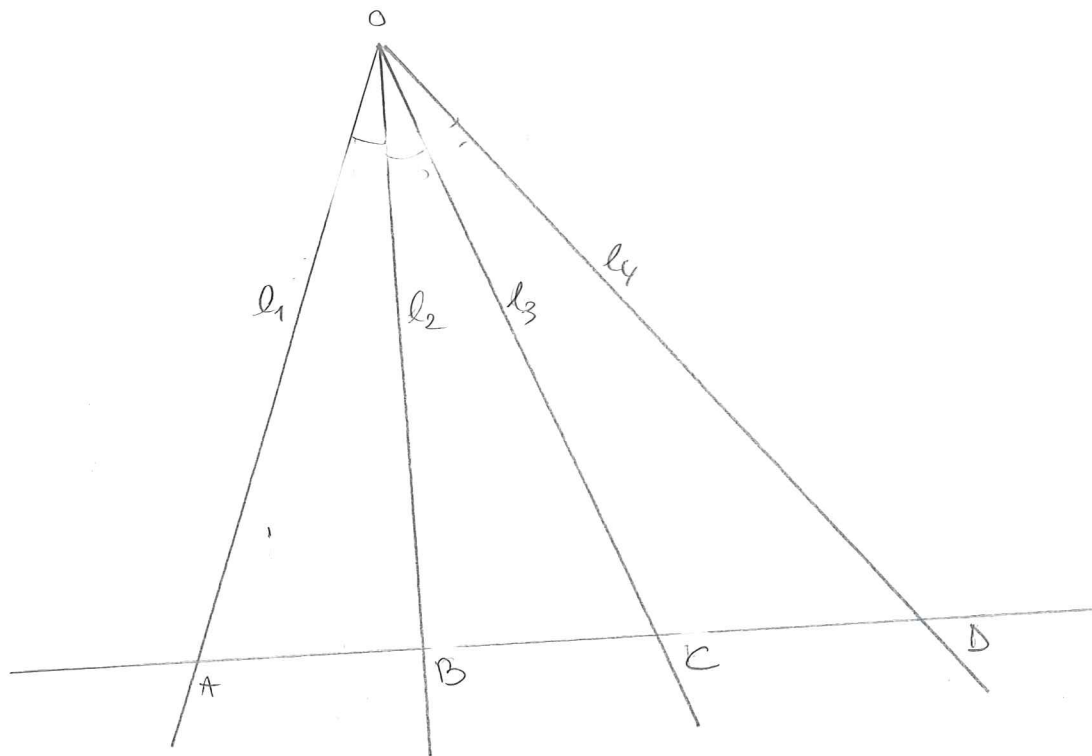
$$\begin{aligned} \text{Kjegle: } u^2 + (v \cos \alpha + 1)^2 - (v \sin \alpha)^2 \\ = u^2 + v^2 \cos^2 \alpha + 2v \cos \alpha + 1 - v^2 \sin^2 \alpha = 0 \end{aligned}$$

$$\text{dvs. } u^2 + \cos 2\alpha v^2 + 2v \cos \alpha + 1 = 0$$

$$\cos 2\alpha = 0 : \text{parabel}$$

$$\cos 2\alpha > 0 : \text{ellipse}$$

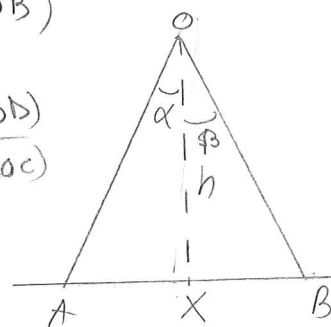
$$\cos 2\alpha < 0 : \text{hyperbel}$$



$$(l_1, l_2; l_3, l_4) = (A, B; C, D) \\ = \frac{\sin(\angle AOC)}{\sin(\angle AOB)} \cdot \frac{\sin(\angle BOD)}{\sin(\angle BOC)}$$

for ΔAOB : $\text{area}(\Delta AOB) = \frac{1}{2} h \cdot |AB| = \frac{1}{2} |OA| \cdot |OB| \cdot \sin(\angle AOB)$

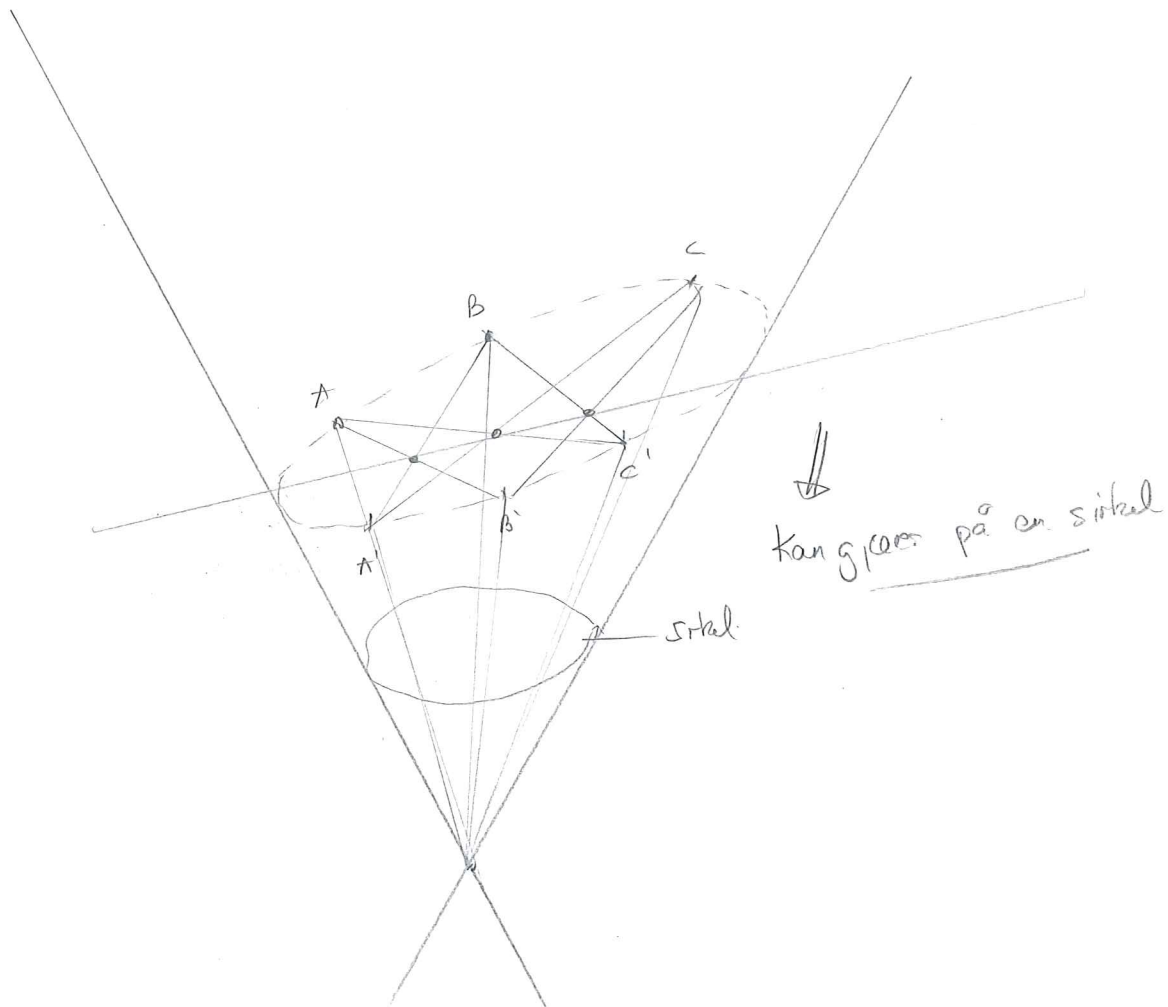
$$\frac{|AC|}{|AD|} \cdot \frac{|BD|}{|BC|} = \frac{\frac{1}{2} |OA| \cdot |OC| \cdot \sin(\angle AOC)}{\frac{1}{2} |OA| \cdot |OD| \cdot \sin(\angle AOD)} \cdot \frac{\frac{1}{2} |OB| \cdot |OD| \cdot \sin(\angle BOD)}{\frac{1}{2} |OB| \cdot |OC| \cdot \sin(\angle BOC)} \\ = \frac{\sin(\angle AOC)}{\sin(\angle AOD)} \cdot \frac{\sin(\angle BOD)}{\sin(\angle BOC)}$$



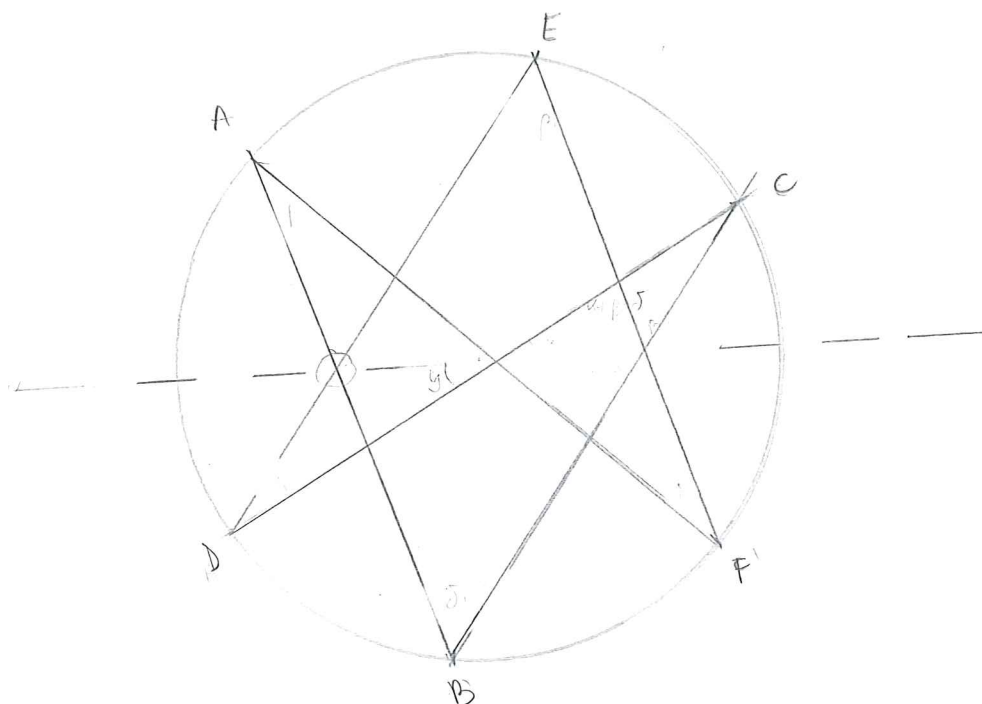
$$\text{area} = \left(|AX| \cdot h + |XB| \cdot h \right) \frac{1}{2} \\ = \frac{1}{2} |OA| \sin \alpha \cdot |OB| \cos \beta \\ + \frac{1}{2} |OB| \sin \beta \cdot |OA| \cos \alpha \\ = \frac{1}{2} |OA| \cdot |OB| \cdot \sin(\alpha + \beta)$$

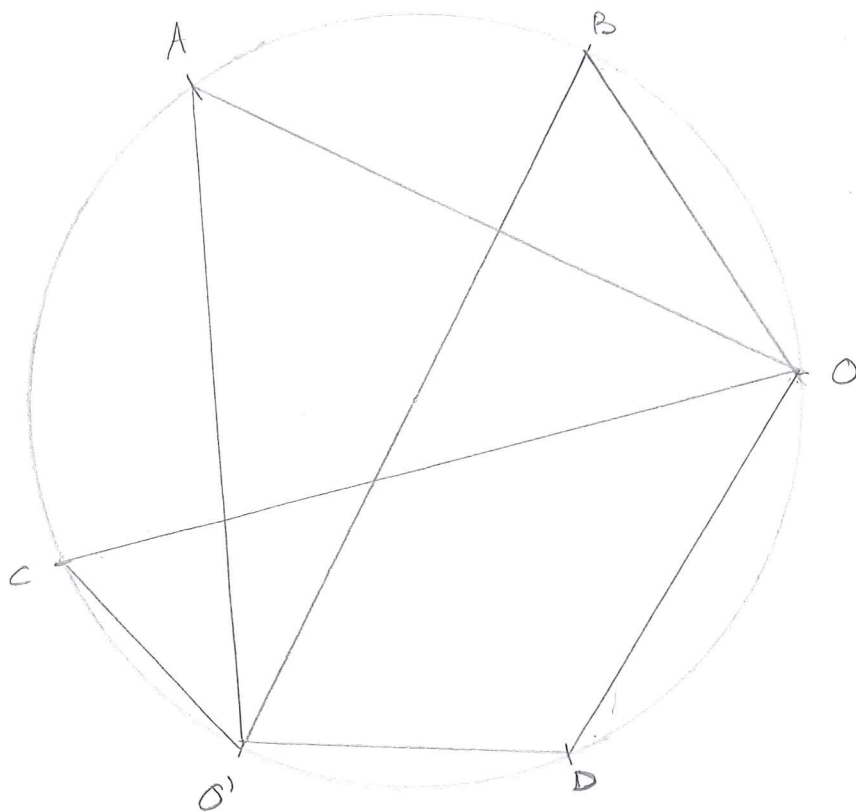


Pascal



Pascal for sirkel





$$\begin{aligned}
 (OA, OB; OC, OD) &= \frac{\sin(\angle AOC)}{\sin(\angle AOB)} \cdot \frac{\sin(\angle BOC)}{\sin(\angle BOD)} \\
 &= \frac{\sin(\angle AO'C)}{(-\sin(\angle AO'D))} \cdot \frac{\sin(\angle BO'C)}{(-\sin(\angle BO'D))} = (O'A, O'B; O'C, O'D)
 \end{aligned}$$

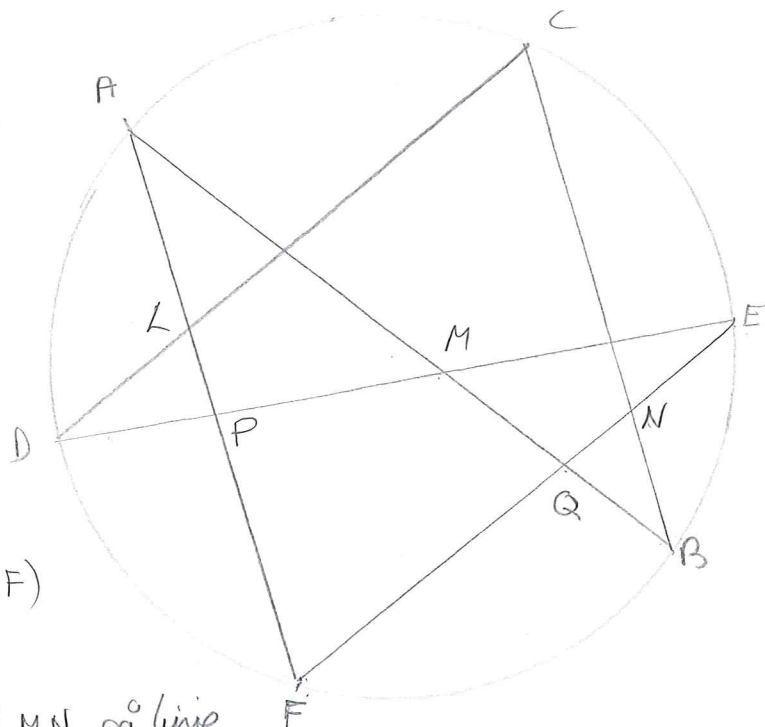
$$\begin{aligned}
 (DA, DC; DE, DF) \\
 = (A, L; P, F) \\
 (BA, BC; BE, BF) \\
 = (Q, N; E, F)
 \end{aligned}$$

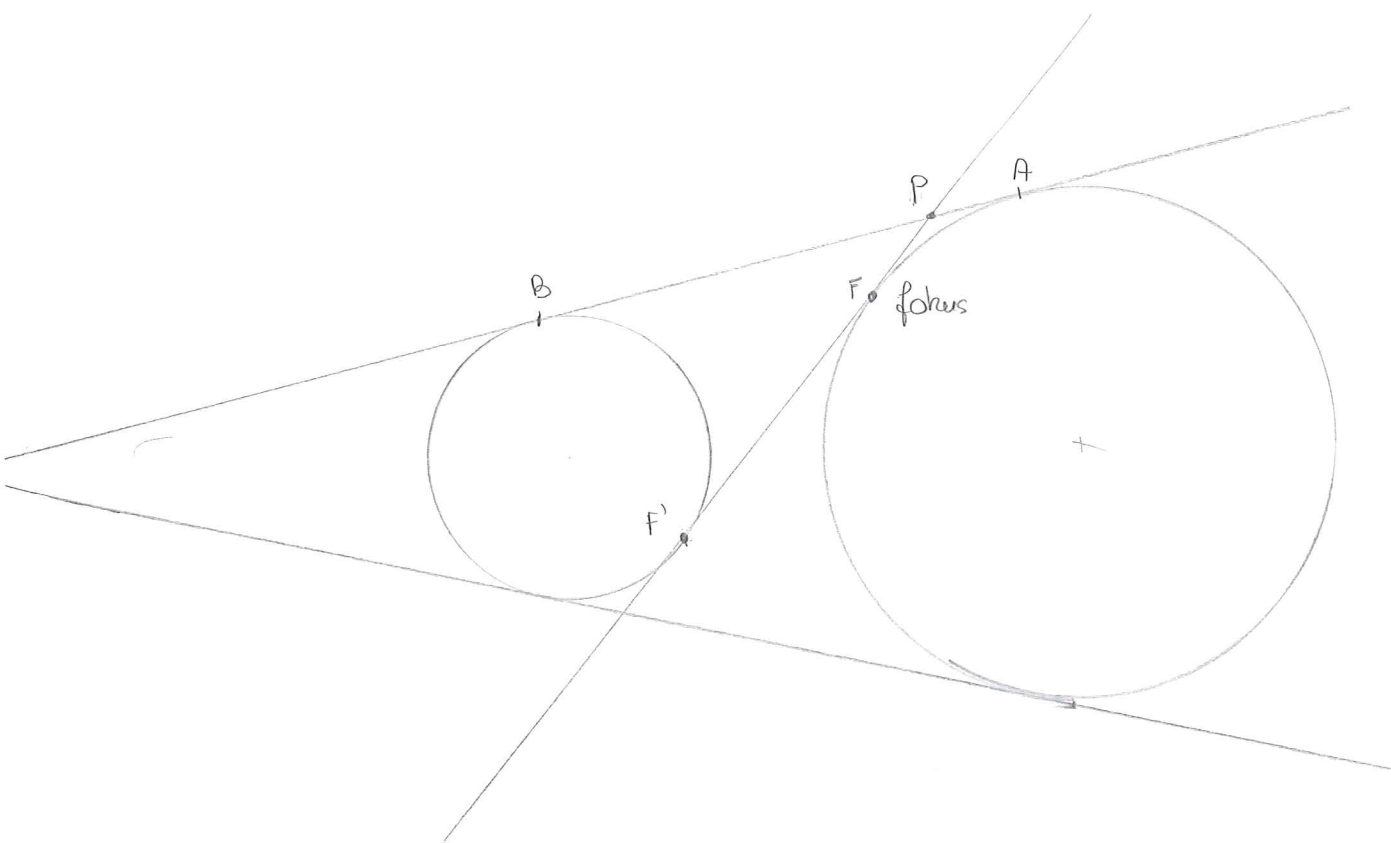
L, M, N på linje



$$(A, L; P, F) = (Q, N; E, F)$$

$\Rightarrow \underline{\underline{L, M, N \text{ på linje}}}$





Dandelin's kuler.

$$PF = PA$$

$$PF' = PB$$

$$\Rightarrow PF + PF' = PA + PB = AB$$

uavhengig av
valg av P p^o
planet.

$\Rightarrow$ ellipse