

MAT 2500 25.10.2022

DUALITET

$$\{\text{linjer i } \mathbb{P}^2\} \mapsto \{\text{Punkter i } (\mathbb{P}^2)^\vee\}$$

$$l: ax_0 + bx_1 + cx_2 = 0 \quad (a:b:c) \in \mathbb{P}^2$$

VIKTIG KONSTRUKSJON:

Velg to pkt i $\mathbb{P}^2$

$$P = (a_0 : a_1 : a_2)$$

$$Q = (b_0 : b_1 : b_2)$$

$$l_{PQ}: ax_0 + bx_1 + cx_2 = 0$$

$$aa_0 + ba_1 + ca_2 = 0$$

$$ab_0 + bb_1 + cb_2 = 0$$

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \text{homogent lin. syst.}$$

$$\det M \neq 0 \Rightarrow (a:b:c) = (a_1b_2 - a_2b_1 : a_2b_0 - a_0b_2 : a_0b_1 - a_1b_0)$$

$$0 = \det \begin{pmatrix} x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} = (a_1b_2 - a_2b_1)x_0 + (a_2b_0 - a_0b_2)x_1 + (a_0b_1 - a_1b_0)x_2$$

$$l_{PQ}: \begin{vmatrix} x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} = 0$$

kan v. forenkles
s.a. $\gcd(a,b,c) = 1$

MÅTSATT:

 $\mathbb{P}^2$
 $(\mathbb{P}^2)^\vee$

$$P = (a_0 : a_1 : a_2) \longleftrightarrow l_P: a_0y_0 + a_1y_1 + a_2y_2 = 0$$

$$Q = (b_0 : b_1 : b_2) \longleftrightarrow l_Q: b_0y_0 + b_1y_1 + b_2y_2 = 0$$

 $(\mathbb{P}^2)^\vee:$

$$l_P \cap l_Q: \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{Her løsning} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1b_2 - a_2b_1 \\ a_2b_0 - a_0b_2 \\ a_0b_1 - a_1b_0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = R = (l_{PQ})^\vee$$

I praksis:

$$\vec{y} = (y_0, y_1, y_2) = (a_0, a_1, a_2) \times (b_0, b_1, b_2) \in \mathbb{R}^3$$

$$\text{og velges representant } (y_0 : y_1 : y_2) \in \mathbb{P}^2$$

EKS 6.6.

$$a) \quad l_1: x_0 - 2x_1 + 3x_2 = 0$$

$$l_2: 3x_0 + x_1 + 4x_2 = 0$$

that plane $P = l_1 \cap l_2$:

$$\begin{aligned} \vec{v}_P &= (1, -2, 3) \times (3, 1, 4) \\ &= (-2 \cdot 4 - 3 \cdot 1, 3 \cdot 3 - 1 \cdot 4, 1 \cdot 1 - (-2) \cdot 3) \\ &= (-11, 5, 7) \end{aligned}$$

$$P = \underline{\underline{(-11:5:7)}}$$

$$b) \quad Q = (0:1:0)$$

Skil plane l_{PQ} .

$$l_{PQ}: \det \begin{pmatrix} x_0 & x_1 & x_2 \\ -11 & 5 & 7 \\ 0 & 1 & 0 \end{pmatrix} = 0$$

$$-7x_0 - 11x_2 = 0$$

$$\underline{\underline{7x_0 + 11x_2 = 0}}$$

EKS a)

$$l_1: x_0 + 2x_2 = 0$$

$$l_2: 2x_0 + x_2 = 0$$

$$\begin{aligned} \vec{v}_P &= (1, 0, 2) \times (2, 0, 1) \\ &= (0, -3, 0) \end{aligned}$$

$$P = \underline{\underline{(0:1:0)}} (= (0:-3:0))$$

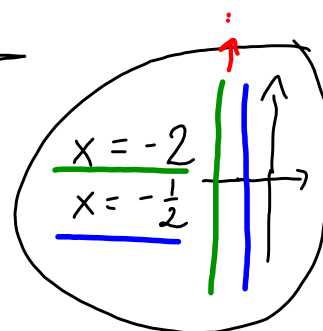
$$b) \quad \text{Gitt } Q = (1:0:0)$$

Hva er l_{PQ} ?

$$l_{PQ}: \begin{vmatrix} x_0 & x_1 & x_2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

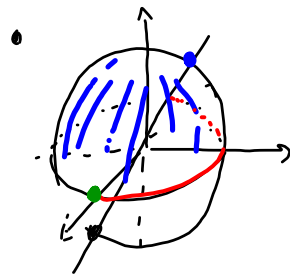
$$\underline{\underline{x_2 = 0}}$$

Se på P og Q .



HVORDAN "SE" $\mathbb{P}^2$?

- $(\mathbb{R}^3 \setminus \{0\}) / \sim$ like hemisfærmengde



$$x_2 > 0$$

$$x_2 = 0 \text{ og } x_1 > 0$$

$$(1, 0, 0)$$

Kanskentlig å få oversikt...

- da $U_i = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : x_i \neq 0\}$ for $i=0,1,2$.

U_0, U_1, U_2 er tre mengder som tilsammen inneholder alle punkter i $\mathbb{P}^2$

U_0, U_1 og U_2 dekker $\mathbb{P}^2$.

- Se på U_2 , der $x_2 \neq 0$

Definer funksjon $U_2 \rightarrow \mathbb{R}^2$

$$(x_0 : x_1 : x_2) \mapsto \left(\frac{x_0}{x_2}, \frac{x_1}{x_2} \right) = (x, y)$$

① veldefinert

$$(\lambda x_0 : \lambda x_1 : \lambda x_2) \mapsto \left(\frac{\lambda x_0}{\lambda x_2}, \frac{\lambda x_1}{\lambda x_2} \right) = \left(\frac{x_0}{x_2}, \frac{x_1}{x_2} \right)$$

Unavh. av representant for punktet

② Injektiv

$$\left(\frac{x_0}{x_2}, \frac{x_1}{x_2} \right) = \left(\frac{y_0}{y_2}, \frac{y_1}{y_2} \right)$$

$$\text{Da er } (y_0 : y_1 : y_2) = \left(\frac{x_0}{x_2} y_2, \frac{x_1}{x_2} y_2, \frac{x_2}{x_2} y_2 \right)$$

$$= \frac{y_2}{x_2} (x_0 : x_1 : x_2)$$

$$= (x_0 : x_1 : x_2)$$

③ Surjektiv $(x : y : 1) \mapsto (x, y)$

KONKLUSJON:

$$U_2 \cong \mathbb{R}^2 \quad \text{På } U_2 \text{ er } x_2 \neq 0$$

$$\lambda \infty : x_2 = 0$$

IVA MED UNDERMENGER?

- $l \subseteq \mathbb{P}^2$ $l: ax_0 + bx_1 + cx_2 = 0$
 $\hat{l} = l \cap U_2 \subseteq \mathbb{R}^2$ $l \cap U_2: a \cdot \frac{x_0}{x_2} + b \cdot \frac{x_1}{x_2} + c = 0$
 $ax + by + c = 0$

- Fra U_2 til $\mathbb{P}^2$:

$$\left. \begin{array}{l} \hat{l}_1: \underline{ax + by + c_1 = 0} \\ \hat{l}_2: ax + by + c_2 = 0 \end{array} \right\} \text{To parallelle linjer i } \mathbb{R}^2_{U_2}$$

$\mathbb{P}^2$:

$$l_1: ax_0 + bx_1 + c_1x_2 = 0$$

$$l_2: ax_0 + bx_1 + c_2x_2 = 0$$

$$\vec{v}_p = (a, b, c_1) \times (a, b, c_2)$$

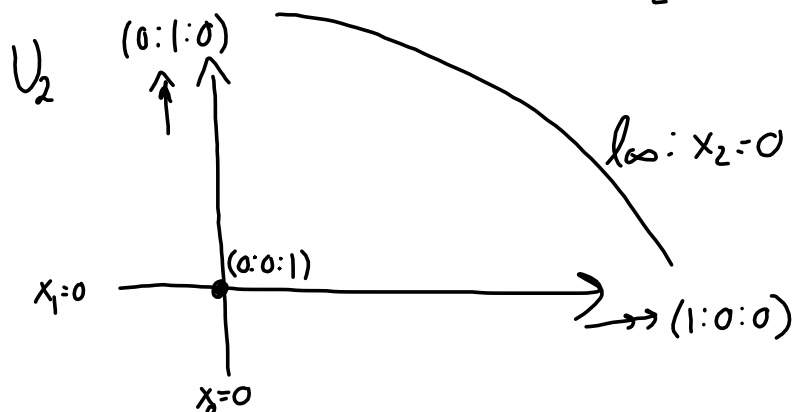
$$= (bc_2 - bc_1, ac_1 - ac_2, ab - ab)$$

$$= (b(c_2 - c_1), -a(c_2 - c_1), 0) \quad c_2 \neq c_1 \text{ siden } \hat{l}_1 \neq \hat{l}_2$$

$$l_1 \cap l_2 = \underline{p = (b : -a : 0)} \quad (= (1 : -\frac{a}{b} : 0) \text{ hvis } b \neq 0)$$

Prøv: $l_1 \cap l_\infty$

$l_2 \cap l_\infty$



H14-4

a) i)

$$A = (1:1:0) \quad B = (0:1:1)$$

Skal finne L_{AB}

$$L_{AB}: 0 = \begin{vmatrix} x_0 & x_1 & x_2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = x_0 - x_1 + x_2$$

$$\text{Sjekk: } 1 - 1 + 0 = 0 \quad \checkmark \quad A \in L_{AB}$$

$$0 - 1 + 1 = 0 \quad \checkmark \quad B \in L_{AB}$$

ii) Skal finne skjæringspnt mellom L_{AB}

$$\text{og } C: x_0^2 + x_0 x_1 - x_1 x_2 - 7x_2^2 = 0$$

$$\begin{pmatrix} U_0: 1 + x - xy - 7y^2 = 0 \\ U_1: x^2 + x - y - 7y^2 = 0 \\ U_2: x^2 + xy - y - 7 = 0 \end{pmatrix} \begin{array}{l} x_0 = 1 \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \\ x_1 = 1 \mapsto \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) \\ x_2 = 1 \mapsto \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) \end{array}$$

$$L_{AB}: x_0 - x_1 + x_2 = 0$$

OBS: x_1 er lineær i likn for C:

$$\text{Velger } x_1 = x_0 + x_2 \quad (L_{AB})$$

Substituerer inn i likn for C:

$$x_0^2 + x_0(x_0 + x_2) - (x_0 + x_2)x_2 - 7x_2^2 = 0$$

$$2x_0^2 - 8x_2^2 = 0$$

$$\text{Se på } U_2: \\ \underline{x_2 = 1}$$

$$2x_0^2 - 8 = 0$$

$$\underline{x_0 = \pm 2}$$

$$P_1 = (2:3:1)$$

$$P_2 = (-2:-1:1)$$

$$U_0: 2 - 8x_2^2 = 0 \\ x_0 = 1 \quad x_2 = \pm \frac{1}{2}$$

$$P_1 = (1: \frac{3}{2}: \frac{1}{2})$$

$$P_2 = (1: \frac{1}{2}: -\frac{1}{2})$$

(U₁ gir også, men
en annen substitusjon)

EKS 6.7. FANOPLANET

Vi kan aksiomatisk definere et projektivt plan
over vilkårlige kroppar ^{primtall}

$$\left\{ \begin{array}{l} \mathbb{Z}_2 = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \\ 0+0=0 \quad 0 \cdot 0=0 \\ 0+1=1 \quad 0 \cdot 1=0 \\ 1+1=0 \quad 1 \cdot 1=1 \end{array} \right.$$

$$\mathbb{P}^2 = (\mathcal{P}, \mathcal{L})$$

↑
Punkter

↑
linjer

$$(l \in \mathcal{L}) \ni \mathcal{P}$$

Krav:

$$|\mathcal{L}| \geq 2 \quad \text{og for alle } l \in \mathcal{L} \\ \text{m} \quad |l| \geq 3$$

• Minst to linjer

• Alle linjer m} ha minst tre punkter.

Aksiommer:

- ① Hvis P og Q er forskjellige pkt, s} er
der en entydig linje l s} $P \in l$ og $Q \in l$.
- ② Hvis l og l' er i $\mathcal{L}$, s} er $l \cap l' \neq \emptyset$

$\mathbb{P}_{\mathbb{F}_2}^2$

$$\mathcal{P} = \left\{ \begin{array}{l} (1:0:0), (0:1:0), (0:0:1) \\ (0:1:1), (1:0:1), (1:1:0), (1:1:1) \end{array} \right\}$$

7 pkt

