

Projektive transformasjoner

Awbildning $T: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ som tar linjer på linjer er gitt ved lineærabildninger $\tilde{T}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\tilde{T}: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad (x:y:z) \in \mathbb{P}^2$$

$$T(x:y:z) = \tilde{T} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Vel-definert siden

$$\tilde{T} \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} = k \tilde{T} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (\text{samme punkt i } \mathbb{P}^2)$$

! Viktig at T er en lineærabildning.!

[se eget ark]

Setning $\mathbb{R}^3$: Gitt to sett av tre ^{ulike} punkter $\neq \emptyset$. Da finnes en ^{entydig} lineærabildning som tar det ene settet på det andre (husk at origo $\mapsto$ origo)

dvs. 4 punkter
Setning $\mathbb{P}^2$: Gitt to sett av 4 ulike punkter i generell posisjon i $\mathbb{P}^2$. Da finnes en entydig projektiv awb (dvs T) som tar det ene settet på det andre

Bevis. Ved å velge basis kan vi la de 4 punktene være $(1:0:0), (0:1:0), (0:0:1)$ og $(1:1:1)$

$$\begin{aligned} (x:y:z) &= x(1:0:0) + y(0:1:0) + z(0:0:1) \\ &= x \left(\frac{1}{x} : 0 : 0 \right) + y \left(0 : \frac{1}{y} : 0 \right) + z \left(0 : 0 : \frac{1}{z} \right) = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right) \end{aligned}$$

De 4 andre punktene er $P' = (x_0:x_1:x_2), Q' = (y_0:y_1:y_2), R' = (z_0:z_1:z_2)$ og $S' = \lambda P' + \mu Q' + \nu R'$

$$\tilde{T} = \begin{pmatrix} \lambda x_0 & \mu y_0 & \nu z_0 \\ \lambda x_1 & \mu y_1 & \nu z_1 \\ \lambda x_2 & \mu y_2 & \nu z_2 \end{pmatrix}$$

gjør jobben.

Entydighet. La $U: (1:0:0) \mapsto (1:0:0), (0:1:0) \mapsto (0:1:0)$
 $\sigma_y (0:0:1) \mapsto (0:0:1)$

$$\Rightarrow [U] = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

$$\Rightarrow U(1:1:1) = (\lambda:\mu:\nu) = (1:1:1)$$

hvis og bare hvis $\lambda = \mu = \nu$

For to projective transformasjoner T_1 og T_2 er
 $T_1 \circ T_2 = T_2 \circ T_1$

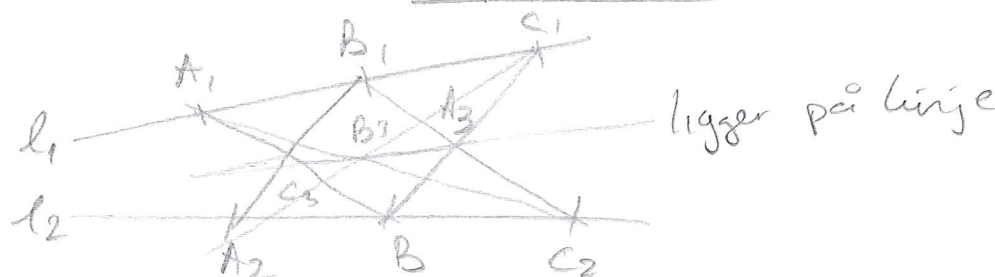
proj. trans som tar P, Q, R, S på seg selv

$$\Rightarrow T_1^{-1} \circ T_2 = \text{id.}$$

$$\Rightarrow T_1 = T_2$$

□

Eksempel på bruk: Pappas i $\mathbb{P}^2$



Bevis: Ved projectiv transformasjon:

$$l_1: X_1 = 0 \quad l_2: X_2 = 0$$

$$\Rightarrow \begin{matrix} \text{Kom anta} & A_1 = (a_1:0:1) & B_1 = (b_1:0:1) & C_1 = (c_1:0:1) \\ & A_2 = (a_2:1:0) & B_2 = (b_2:1:0) & C_2 = (c_2:1:0) \end{matrix}$$

$$A_3 = (b_1b_2 - c_1c_2 : b_1 - c_1 : b_2 - c_2)$$

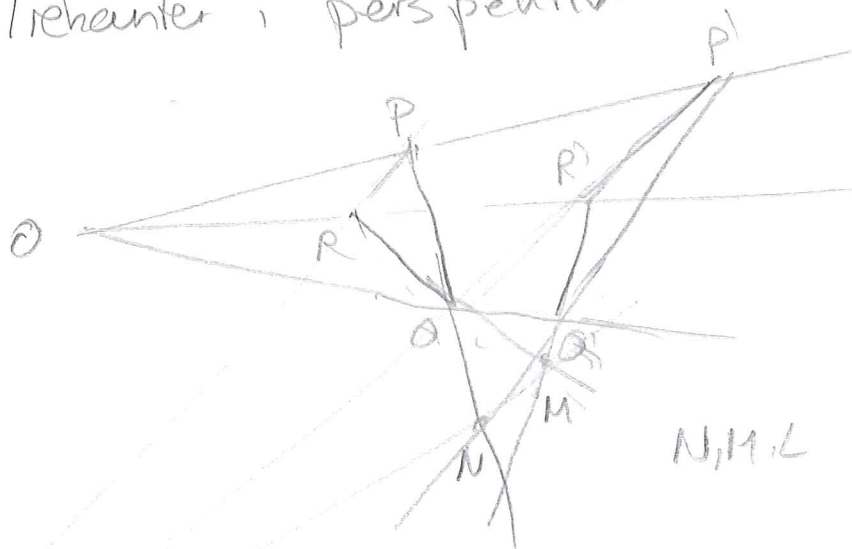
$$B_3 = (c_1c_2 - a_1a_2 : c_1 - a_1 : c_2 - a_2) \quad \text{i } \mathbb{R}^3$$

$$C_3 = (a_1a_2 - b_1b_2 : a_1 - b_1 : a_2 - b_2)$$

$$(0,0,0)$$

$\Rightarrow$ Punktene ligger i
 Samme plan.

Trekanter i perspektiv.



N, M, L kollinear

Basis $O = (1:1:1)$ $P = (1:0:0)$
 $Q = (0:1:0)$
 $R = (0:0:1)$

$$\Rightarrow P' = (p:1:1) \quad Q' = (1:q:1) \quad R' = (1:1:r)$$

$$PQ = \begin{vmatrix} x_0 & x_1 & x_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = x_2 = 0$$

$$P'Q' = \begin{vmatrix} x_0 & x_1 & x_2 \\ p & 1 & 1 \\ 1 & q & 1 \end{vmatrix} = (1-q)x_0 + (1-p)x_1 + (pq-1)x_2$$

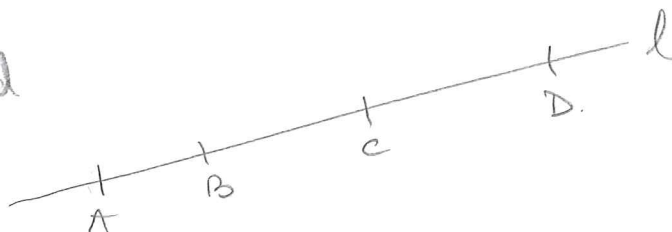
$$\Rightarrow N = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1-q & 1-p & pq-1 \end{vmatrix} = (-1+p:1-q:0)$$

Tilsvarende: $L = (-1+p:0:1-r)$, $M = (0:1-q:r-1)$

Som vektorer $(p-1, 1-q, 0) + (1-p, 0, r-1) + (0, q-1, 1-r) = \underline{\underline{0}}$

$\Rightarrow$ ligger på linje

Kryssforhold



$$(AB, CD) = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}}$$

Harmoniske konjugert

punkter: $(AB, CD) = -1$

Anta l er x-aksen, $A=0,0$ $B=(b,0)$ $C=(c,0)$ $D=(d,0)$

Da har vi

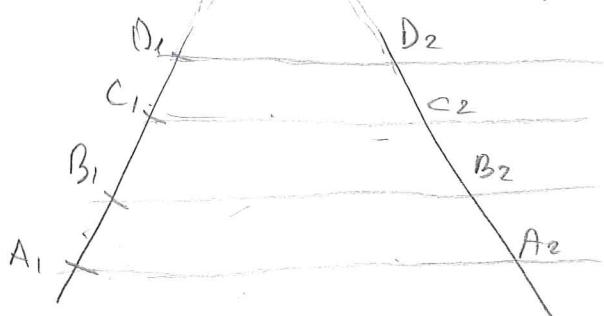
$$(AB, CD) = \frac{c}{d} \frac{d-b}{c-b} = \frac{cd-bc}{cd-bd}$$

Lineaarbeildning $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$: $A \rightarrow A$, $B \rightarrow b \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ $C \mapsto c \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$
 $D \mapsto d \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$

$$AC = c \sqrt{x_{11}^2 + x_{21}^2} \quad AB = b \sqrt{x_{11}^2 + x_{21}^2}$$

$$BD = (d-b) \sqrt{x_{11}^2 + x_{21}^2} \quad BC = (c-b) \sqrt{x_{11}^2 + x_{21}^2}$$

$\Rightarrow$ Kryssforhold er bevart



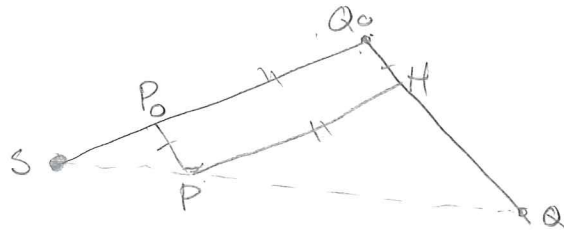
$$\frac{C_1 C_2}{A_1 A_2} = \frac{PC_1}{PA_1} = \frac{PC_2}{PA_2} \Rightarrow \frac{PC_1}{PC_1 + A_1 C_1} = \frac{PC_2}{PC_2 + A_2 C_2}$$

$$\Rightarrow PC_1 \cdot PC_2 + PC_1 A_2 C_2 = PC_1^2 + PC_2^2 + PC_2 A_1 C_1$$

$$\Rightarrow \frac{PC_1}{A_1 C_1} = \frac{PC_2}{A_2 C_2} = \text{ell} = \frac{A_1 C_1}{A_2 C_2} = \frac{PC_1}{PC_2}$$

$$\Rightarrow \frac{A_1 C_1}{A_2 C_2} \cdot \frac{A_2 D_2}{A_1 D_1} \cdot \frac{B_1 D_1}{B_2 D_2} \cdot \frac{B_2 C_2}{B_1 C_1} = \frac{PC_1}{PC_2} \cdot \frac{PD_2}{PD_1} \cdot \frac{PD_1}{PD_2} \cdot \frac{PC_2}{PC_1} = 1$$

Pantograf



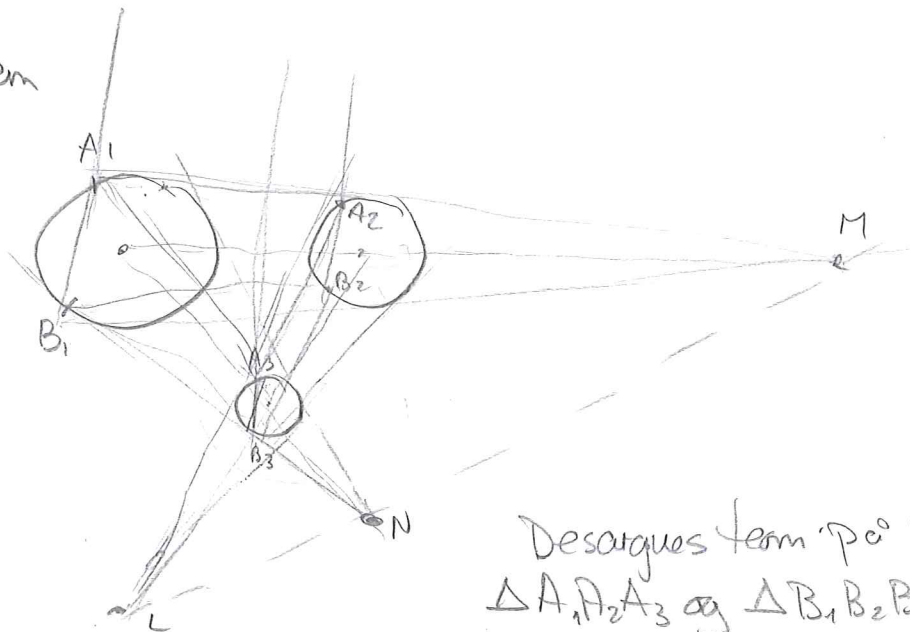
$$\begin{aligned} SQ_0 &= k \cdot SP_0 & HQ_0 &= PP_0 \\ PH &= P_0 Q_0 & QQ_0 &= kQ_0 H \end{aligned}$$

$\Rightarrow S, P, Q$ ligger på linje

$$\frac{Q Q_0}{P P_0} = \frac{k Q_0 H}{P P_0} = \frac{k \cdot P P_0}{P P_0} = k$$

$$\frac{SQ_0}{SP_0} = k$$

Monges leonem



Desargues term. P^0 Hechante
 $\Delta A_1 A_2 A_3 \sim \Delta B_1 B_2 B_3$.

$$\Rightarrow A_1 A_2 \cap B_1 B_2, A_1 A_3 \cap B_1 B_3, A_2 A_3 \cap B_2 B_3$$

$$\quad \quad \quad \parallel \quad \quad \quad \parallel \quad \quad \quad \parallel$$

$$\quad \quad \quad M \quad \quad \quad N \quad \quad \quad L$$

Rolüneare