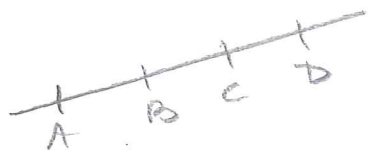


Kryss forhold



$$(AB, CD) = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}}$$

$(AB, CD) = -1$: Punktene er harmonisk konjugerte

Harmonisk deling

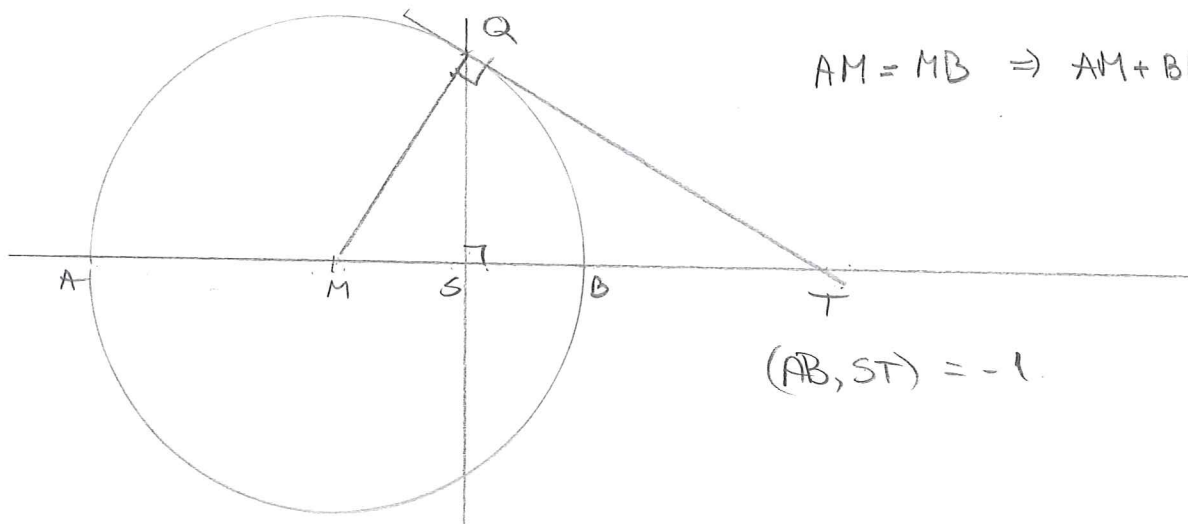
$$\frac{1}{AB} = \frac{1}{2} \left(\frac{1}{AC} + \frac{1}{AD} \right) \Leftrightarrow 2 = \frac{\overline{AB}}{\overline{AC}} + \frac{\overline{AB}}{\overline{AD}}$$

$$2 = \frac{\overline{AC} + \overline{CB}}{\overline{AC}} + \frac{\overline{AD} + \overline{DB}}{\overline{AD}}$$

$$\Leftrightarrow \frac{\overline{CB}}{\overline{AC}} + \frac{\overline{DB}}{\overline{AD}} = 0 \quad \text{eller} \quad \frac{\overline{CB}}{\overline{AC}} = -\frac{\overline{DB}}{\overline{AD}}$$

$$\Leftrightarrow \frac{\overline{AC}}{\overline{AB}} \cdot \frac{\overline{BD}}{\overline{BC}} = -1$$

Konstruksjon



$$AM = MB \Rightarrow AM + BM = 0$$

$$(AB, ST) = -1$$

$$\triangle MSQ \sim \triangle MQT \Rightarrow \frac{MS}{MQ} = \frac{MQ}{MT} \Rightarrow MS \cdot MT = MQ^2$$

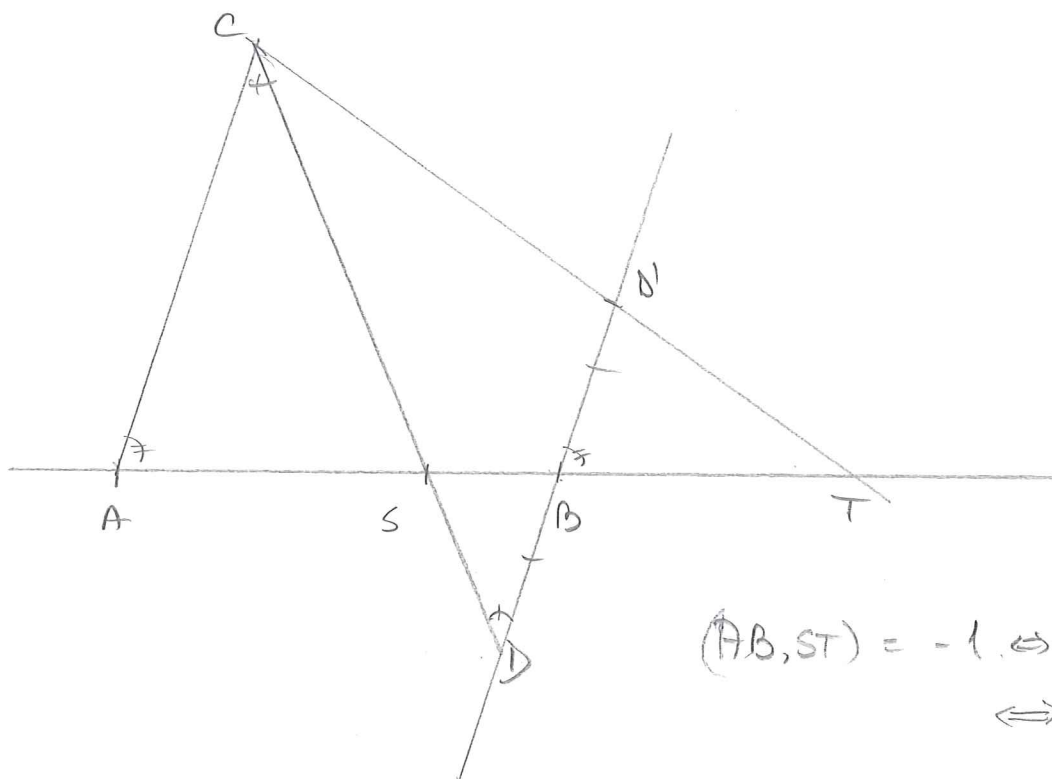
$$\Rightarrow MS \cdot MT = AM \cdot MB = -MA \cdot MB$$

$$AS \cdot BT + AT \cdot BS = (AM + MS)(BM + MT) + (AM + MT)(BM + MS)$$

$$= \cancel{AM \cdot BM} + AM \cdot MT + MS \cdot BM + \cancel{MS \cdot MT} + \cancel{AM \cdot BM} + AM \cdot MS + \cancel{MT \cdot BM} + \cancel{MT \cdot MS}$$

$$= AM \cdot MT + AM \cdot MS + MS \cdot BM + MT \cdot BM$$

$$= (AM + BM)(MT + MS) = 0$$



$$(AB, ST) = -1 \Leftrightarrow \frac{AS}{AT} \cdot \frac{BT}{BS} = -1$$

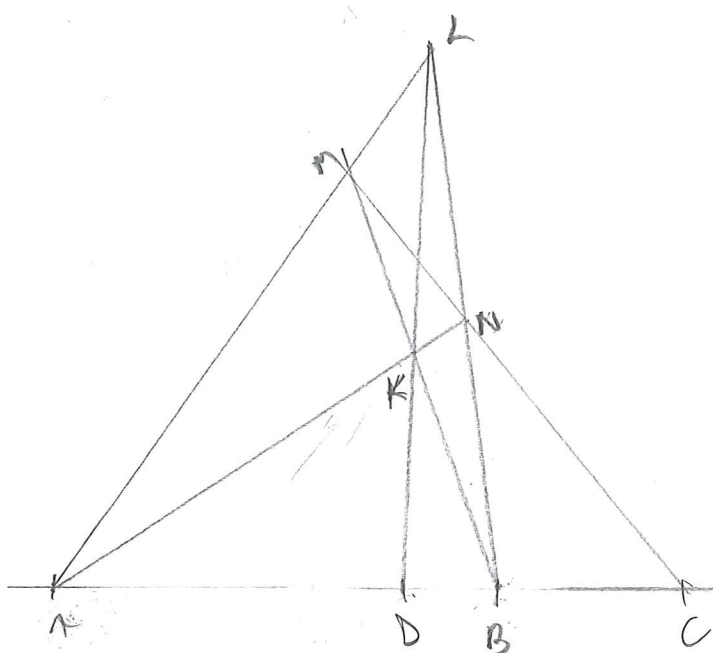
$$\Leftrightarrow \frac{AS}{BS} = -\frac{AT}{BT}$$

$$\frac{AS}{SB} = \frac{AC}{CD}$$

$$\frac{AT}{BT} = \frac{AC}{CD} = \frac{AC}{BD}$$

$$\Rightarrow \frac{AS}{SB} = \frac{AT}{BT} \Rightarrow \frac{AS}{BS} = -\frac{AT}{BT} \quad \square$$

Projektw.



$$D = A + \lambda B$$

$$K = D + \mu L$$

$$\Rightarrow K = A + \lambda B + \mu L$$

$$\Rightarrow K - A = \lambda B + \mu L = N$$

Siden N er skjærsen $AK \cap LB$

$$\text{og } K - \lambda B = A + \mu L = M$$

(samme grunn)

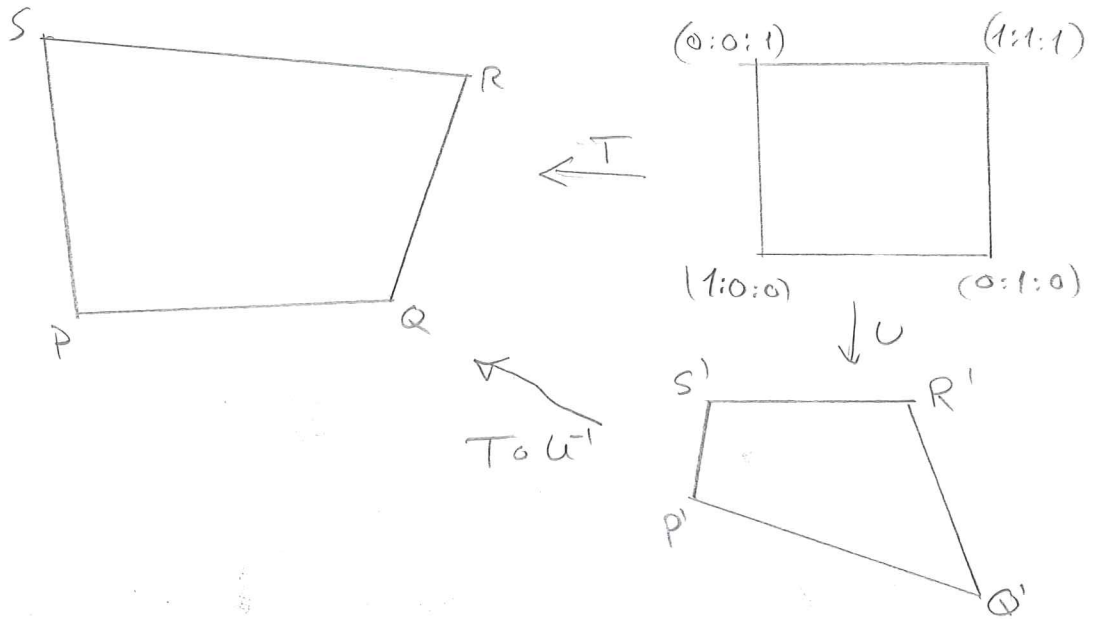
$$\Rightarrow M - N = A - \lambda B = C$$

betyr C er skjærsen mellom MN og AB.

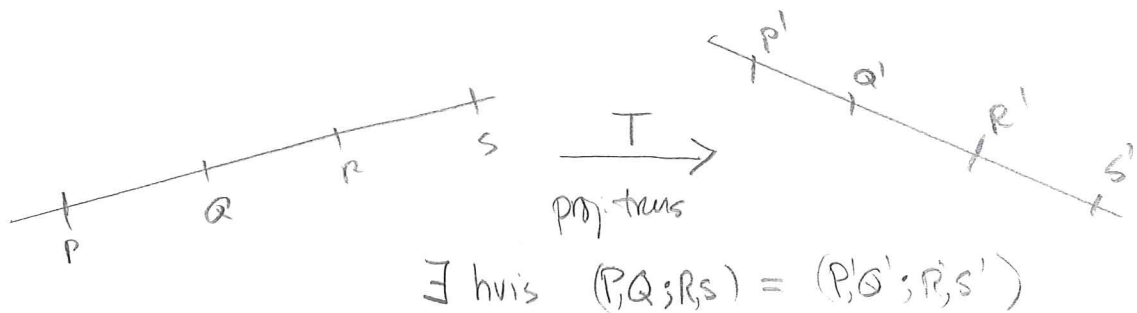
Det harmoniske punktet til $D = A + \lambda B$ er $C = A - \lambda B$

$$\text{eller } D = \frac{1}{1+\lambda} A + \frac{\lambda}{1+\lambda} B, C = \frac{1}{1-\lambda} A - \frac{\lambda}{1-\lambda} B \text{ eller } D - A = \lambda(B - D), C - A = \lambda(C - B)$$

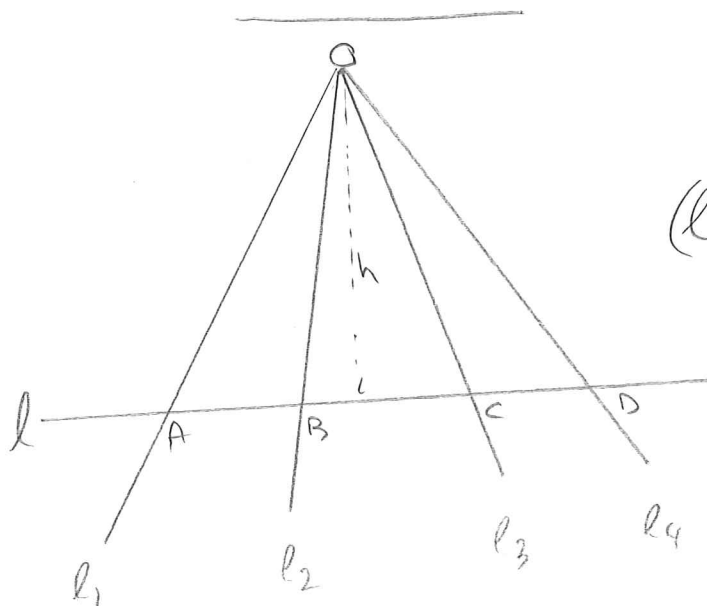
$$\Rightarrow \frac{AC}{AD} \cdot \frac{BD}{BC} = \frac{\lambda(C-B)}{\lambda(B-D)} \cdot \frac{(D-B)}{(C-B)} = -1$$



Hva hvis punktene ligger på linje



$$\exists \text{ hvis } (P, Q; R, S) = (P', Q'; R', S')$$



Definere

$$(l_1, l_2; l_3, l_4)$$

$$= (l_1 \cap l, l_2 \cap l; l_3 \cap l, l_4 \cap l)$$

Veldefineret siden
perspektiv bevarer
kryssprodukt

$$= (A, B; C, D)$$

Vihar

$$(l_1, l_2; l_3, l_4) = \frac{\sin(\angle l_1 l_3)}{\sin(\angle l_1 l_4)} \cdot \frac{\sin(\angle l_2 l_4)}{\sin(\angle l_2 l_3)}$$

Beweis

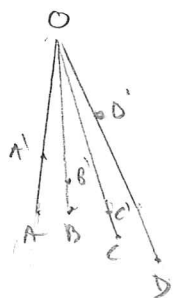
$$\text{Area of } \triangle AOC : \frac{1}{2} \cdot h \cdot AC = \frac{1}{2} OA \cdot OC \cdot \sin(\angle l_1 l_3)$$

$$\Rightarrow \sin(\angle l_1 l_3) = \frac{h \cdot AC}{OA \cdot OC}$$

$$\Rightarrow (l_1, l_2; l_3, l_4) = \frac{\frac{h \cdot AC}{OA \cdot OC}}{\frac{h \cdot AB}{OA \cdot OD}} \cdot \frac{\frac{h \cdot BD}{OB \cdot OD}}{\frac{h \cdot BC}{OB \cdot OC}}$$

$$= \frac{AC}{AD} \cdot \frac{BD}{BC} = (A, B; C, D)$$

A, B, C, D på linje, $\mathbb{P}^2$, $\leadsto$ Representeret et plan, $\mathbb{R}^3$



$$C = pA + qB \quad D = rA + sB$$

$$(A, B; C, D) = \frac{qr}{ps}$$

$$A' = \mu A, B' = \delta B, C' = \nu C, D' = \gamma D$$

$$\Rightarrow C' = \nu C = \nu(pA + qB) = \frac{\nu p}{\mu} A' + \frac{\nu q}{\delta} B'$$

$$D' = \gamma D = \gamma(rA + sB) = \frac{\gamma r}{\mu} A' + \frac{\gamma s}{\delta} B'$$

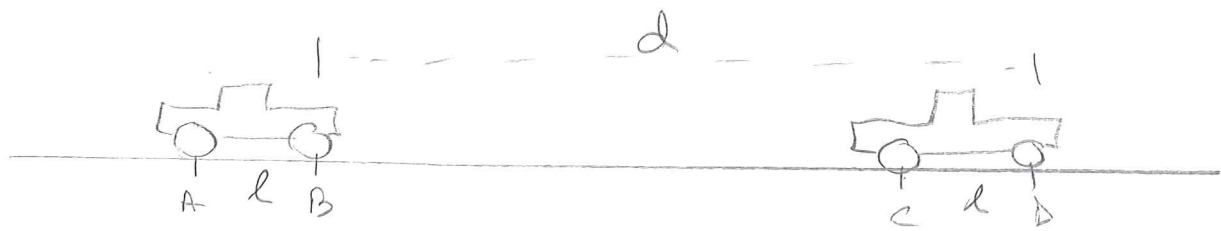
$$(A', B'; C', D') = \frac{\frac{\nu q}{\delta} \frac{\gamma r}{\mu}}{\frac{\nu p}{\mu} \frac{\gamma s}{\delta}} = \frac{qr}{ps}$$

A	B	C	D
a	b	pa+qb	ra+sb
	a+1	p+q=1	r+s=1

$$\frac{AC}{AD} \frac{BD}{BC} = \frac{((p-1)a+qb)(r+(s-1)b)}{((r-1)a+sb)(p+(q-1)b)}$$

$$= \frac{(p+q-1)a+q}{(r+s-1)a+s-1}$$

$$= \frac{q(s-1)}{s(q-1)} = \frac{qr}{sp}$$



Måler i bildet $(A, B; C, D) = K$

i virkeligheten: $(A, B; C, D) = \frac{l}{d+l} : \frac{l}{d-l} = \frac{d^2}{d^2-l^2} = K$

Kryssforhold er bevart ved perspektiv.

$$\Rightarrow K = \frac{d^2}{d^2-l^2} \quad \text{eller} \quad d = \sqrt{\frac{K}{K-1}} \cdot l$$

$$A = (0, 0)$$

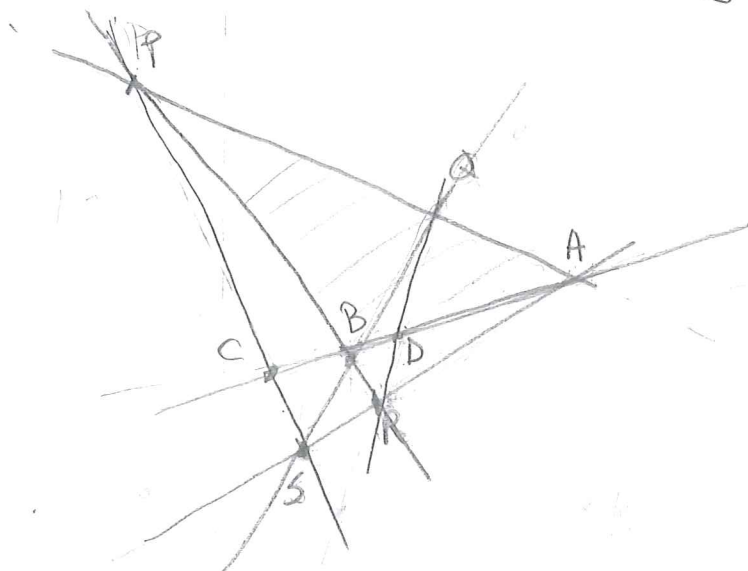
$$B = (3, 0)$$

$$C = (1, 0)$$

Finn D slik at $(A, B; C, D) = -1$.

$$(d, 0)$$

$$-1 = \frac{1}{d} : \frac{d-3}{-2} \Rightarrow 2d = d-3 \Rightarrow \underline{\underline{d = -3}}$$



$\exists P, Q, R, S$ s.t. we can

$$\left. \begin{aligned} PQ \cap RS &= A \\ PR \cap QS &= B \\ PS \cap AB &= C \\ QR \cap AB &= D \end{aligned} \right\}$$



$$\frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}} = -1.$$

Menelaos po $\triangle PAB$: $\frac{\overline{PQ}}{\overline{QA}} \cdot \frac{\overline{AD}}{\overline{DB}} \cdot \frac{\overline{BR}}{\overline{RP}} = -1.$

Ceva po $\triangle PAB$:

$\underbrace{QB, AR, PC}_{\text{Konkurante}}$

$$\frac{\overline{AQ}}{\overline{QP}} \cdot \frac{\overline{PR}}{\overline{RB}} \cdot \frac{\overline{BC}}{\overline{CA}} = 1.$$

$$\Rightarrow \frac{\overline{PQ}}{\overline{QA}} \cdot \frac{\overline{AD}}{\overline{DB}} \cdot \frac{\overline{BR}}{\overline{RP}} \cdot \frac{\overline{AQ}}{\overline{QP}} \cdot \frac{\overline{PR}}{\overline{RB}} \cdot \frac{\overline{BC}}{\overline{CA}} = \frac{\overline{AD}}{\overline{DB}} \cdot \frac{\overline{BC}}{\overline{CA}} = -1$$

$$\Rightarrow \underline{\underline{\frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}} = -1.}}$$