

Kegelschnitt in $\mathbb{P}^2$

Kvadratisch form in 3 variable

$$Q(x_0, x_1, x_2) = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_2^2 \in \mathbb{R}[x_0, x_1, x_2]$$

Homogen

$$Q(kx_0, kx_1, kx_2) = k^2 Q(x_0, x_1, x_2)$$

Setze $x = \frac{x_0}{x_2}$, $y = \frac{x_1}{x_2}$ $x_2 \neq 0$ ∞ : $x_2 = 0$

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Ellipse in $\mathbb{R}^2$ snitten, lke ∞

Parabel in $\mathbb{R}^2$: $ax^2 + dx + ey + f = 0$

$$\Leftrightarrow y = -ax^2 + \frac{d}{e}$$

Homogen $ax_0^2 - x_2x_1 + fx_2^2 = 0$

Snitten $x_2 = 0$ i $x_0 = 0$ dvs $(0:1:0)$

Hyperbel in $\mathbb{R}^2$ $ax^2 - cy^2 + 1 = 0$

Homogen $ax_0^2 - cx_1^2 + x_2^2 = 0$

Snitten $x_2 = 0$ i $\sqrt{a}x_0 + \sqrt{c}x_1 = 0$ o $\sqrt{a}x_0 - \sqrt{c}x_1 = 0$
asymptotene

C: $Q(x_0, x_1, x_2) = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_2^2$

Diskriminant $D_{x_2} = b^2 - 4ac$

i) $D_{x_2} < 0$ $C \cap (x_2 = 0) = \emptyset$ Ellipse

ii) $D_{x_2} = 0$ $C \cap (x_2 = 0) = \{pt\}$ Parabel

iii) $D_{x_2} > 0$ $C \cap (x_2 = 0) = \{p_1, p_2\}$ Hyperbel

Kvadratisk form

$$(x_0 \ x_1) \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

↑
Har egenverdier gitt ved

$$\chi(\lambda) = \lambda^2 - (a+c)\lambda + \left(ac - \frac{b^2}{4}\right) = 0$$

$$\Rightarrow \lambda = \frac{a+c \pm \sqrt{a^2 + 2ac + c^2 - 4ac + b^2}}{2}$$

$$= \frac{a+c \pm \sqrt{(a-c)^2 + b^2}}{2}$$

Determinant: $ac - \frac{b^2}{4} = \frac{1}{4}(b^2 - 4ac) = \lambda_1 \lambda_2$

$\Rightarrow$ i) λ_1, λ_2 har samme tegn $\Leftrightarrow b^2 - 4ac < 0$

ii) λ_1, λ_2 har motsatt tegn $\Leftrightarrow b^2 - 4ac > 0$

iii) $\lambda_1 = 0 \Leftrightarrow b^2 - 4ac = 0$

Eksempel

$$x_0^2 + 2x_0x_1 - 4x_1^2 + ax_1x_2 = 0$$

$$D = 4 - 4 \cdot 1 \cdot (-4) = 20 > 0 \text{ Hyperbel}$$

Forklars

Setning:

$\exists$ proj. transformasjon som tar en ikke-degenerert
kvadratisk til $x_0^2 + x_1^2 - x_2^2 = Q$

Vi kan skrive

$$Q = (x_0 \ x_1 \ x_2) \begin{pmatrix} a & \frac{1}{2}b & \frac{1}{2}d \\ \frac{1}{2}b & c & \frac{1}{2}e \\ \frac{1}{2}d & \frac{1}{2}e & f \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

$$= ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_2^2$$

Orthogonal diagonaliserer vi

$$\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2$$

$$\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = 0$$

$\lambda_0, \lambda_1, \lambda_2 > 0$ Ingen lösning

$\lambda_0, \lambda_1 = 0$ Degenerat linje

$\lambda_0 = 0, \lambda_1, \lambda_2 > 0$ Ett punkt $(1:0:0)$

$\lambda_0 = 0, \lambda_1 > 0, \lambda_2 < 0$ $\lambda_1 x_1^2 - (-\lambda_2) x_2^2 = 0$ Två linjer: $\times$

$\lambda_0 = 0, \lambda_1, \lambda_2 < 0$ Ett punkt $(1:0:0)$

$$\lambda_0 < 0, \lambda_1, \lambda_2 > 0 \quad -(\lambda_0) x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = 0$$

Hyperbel eller ellips

Sättning

$$\begin{matrix} L & C \\ \text{linje,} & \text{quadratisk, } \mathbb{C}P^2 \end{matrix} \Rightarrow L \cap C = \begin{cases} \emptyset \\ \text{Ett} \\ \{P_1, P_2\} \end{cases}$$

Definition

$\{P_1, \dots, P_n\} \Leftrightarrow$ Ingen 3 ligger på en linje
generellt position

Sättning

5 punkter i $\mathbb{P}^2$; generellt position.
bestämmer ett kvadrant.

Beweis

$$P_1 = (1:0:0) \quad P_2 = (0:1:0) \quad P_3 = (0:0:1) \quad P_4 = (1:1:1) \quad P_5 = (y_0:y_1:y_2)$$

$$Q = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_2^2 = 0$$

$$P_1 \rightarrow P_4 \text{ ger } a=c=f=0 \quad b+d+e=0$$

$$P_5 \text{ ger } by_0y_1 + dy_0y_2 + ey_1y_2 = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ y_0y_1 & y_0y_2 & y_1y_2 \end{pmatrix} \begin{pmatrix} b \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow y_0y_1 = y_0y_2 = y_1y_2 = 0$$

$$y_0 = 0 \Rightarrow y_1y_2 = 0 \Rightarrow P_5 = P_2 \text{ eller } P_5 = P_3$$

$$y_0 \neq 0 \text{ og } y_1 = y_2 = 0 \Rightarrow P_5 = P_1$$

$$y_0, y_1 \neq 0 \Rightarrow y_2 \neq 0 \Rightarrow P_5 = P_4$$

unlös $\Rightarrow$ Endelig lösning

Lösung:

5 Punkte:

$(\alpha_i, \beta_i, \gamma_i)$

$$Q = \det \begin{pmatrix} x_0^2 & x_0 x_1 & x_1^2 & x_0 x_2 & x_1 x_2 & x_2^2 \\ \alpha_1^2 & \alpha_1 \beta_1 & \beta_1^2 & \alpha_1 \gamma_1 & \beta_1 \gamma_1 & \gamma_1^2 \\ \alpha_2^2 & \alpha_2 \beta_2 & \beta_2^2 & \alpha_2 \gamma_2 & \beta_2 \gamma_2 & \gamma_2^2 \\ \alpha_3^2 & \alpha_3 \beta_3 & \beta_3^2 & \alpha_3 \gamma_3 & \beta_3 \gamma_3 & \gamma_3^2 \\ \alpha_4^2 & \alpha_4 \beta_4 & \beta_4^2 & \alpha_4 \gamma_4 & \beta_4 \gamma_4 & \gamma_4^2 \\ \alpha_5^2 & \alpha_5 \beta_5 & \beta_5^2 & \alpha_5 \gamma_5 & \beta_5 \gamma_5 & \gamma_5^2 \end{pmatrix}$$

Kvadratisch form, $Q(\alpha_i, \beta_i, \gamma_i) = 0$.