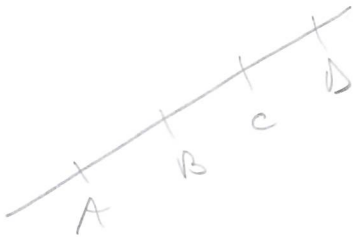
Mandag 19. oktober

Mer om krysstorhold:



$$(AB,CD) = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}}$$

$(A,B),(C,D)$ harmonisk konjugerte:

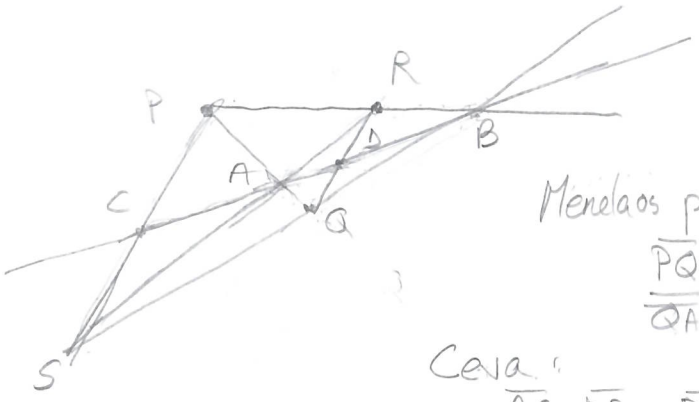
$$(AB,CD) = -1$$

Krysstorhold er en projektiv invariant.

C og D deler linjestykket AB:

$$\frac{\overline{AC}}{\overline{CB}} = -\frac{\overline{AD}}{\overline{DB}}$$

Harmonisk konjugerte



Firhant PRQS:

A = PQ ∩ RS
B = PR ∩ QS
C = PS ∩ AB
D = RQ ∩ AB

Menelaos på $\triangle PAB$

$$\frac{\overline{PQ}}{\overline{QA}} \cdot \frac{\overline{AD}}{\overline{DB}} \cdot \frac{\overline{BR}}{\overline{RP}} = -1$$ (ligger på linje)

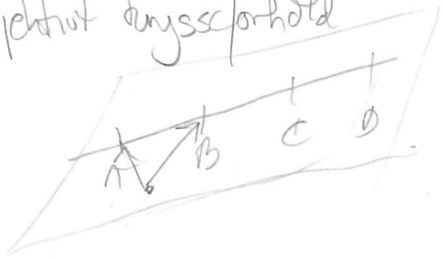
Ceva:

$$\frac{\overline{AC}}{\overline{CB}} \frac{\overline{BR}}{\overline{RP}} \cdot \frac{\overline{PQ}}{\overline{QA}} = 1$$ koeinear

$$\Rightarrow (AB,CD) = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}} = \frac{\overline{AC}}{\overline{AD}} \frac{\overline{BD}}{\overline{BC}} \frac{\overline{CB}}{\overline{AC}} \frac{\overline{RP}}{\overline{BR}} \frac{\overline{QA}}{\overline{PQ}} \frac{\overline{PQ}}{\overline{QA}} \frac{\overline{AB}}{\overline{DB}} \frac{\overline{BR}}{\overline{RP}} (-1)$$

$$= \frac{\overline{BD}}{\overline{BC}} \frac{\overline{CB}}{\overline{DB}} = -1$$

Projektivt krysstorhold



$$C = pA + qB$$
$$D = rA + sB$$

Vi låter $\bar{AB} = B - A$, $\|\bar{AB}\| = 1$.



$\bar{AC} = (p-1)A + qB = qB - qA$

$p - 1 = -q \Rightarrow p + q = 1$.

$\bar{AD} = (r-1)A + sB = sB - sA$.

$r - 1 = -s \Rightarrow r + s = 1$.

$\bar{BC} = pA + (q-1)B = pA - pB$

$\bar{BD} = rA + (s-1)B = rA - sB$

$\Rightarrow (AB, CD) = \frac{q}{s}\frac{r}{p} = \frac{qr}{ps}$

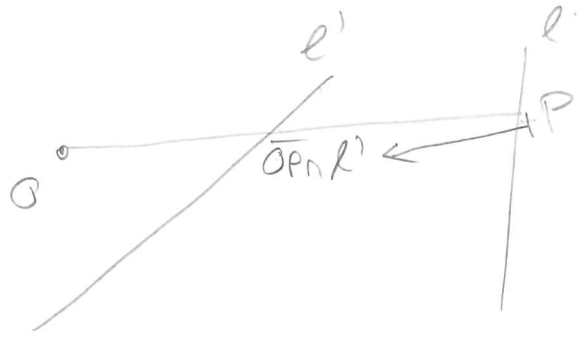


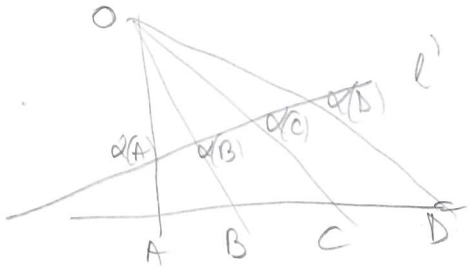
$(AB, CX) = (AB, CY)$

$\Downarrow$

$X = Y$.

Perspektivitet:





$\{\bar{OA}, \bar{OB}, \bar{OC}, \bar{OD}\}$ $\xrightarrow{\text{dualt}}$ $\{P, Q, R, S\}$

linjer — mötes i ett punkt

punkter — ligger på linje

Linje skärvara plan i $\mathbb{R}^3$. Normalvektor $\bar{n}$ till planet $\longrightarrow$ $\bar{n}$ ger dualpunkt

Dualitet

$P = (x_0 : x_1 : x_2)$ punkt svarer til $\ell : x_0 x + x_1 y + x_2 z = 0$

Linje gjennom P og $Q = (y_0 : y_1 : y_2)$ — $(x:y:z)$ skjæring mellom ℓ og $y_0 x + y_1 y + y_2 z = 0$

$$\begin{vmatrix} x & y & z \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix} = 0 \qquad (x_0\ x_1\ x_2)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (y_0\ y_1\ y_2)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Leftrightarrow \quad x(x_1y_2 - x_2y_1) + y(x_2y_0 - x_0y_2) + z(x_0y_1 - x_1y_0) = 0$$

$$\updownarrow \text{ svarer til}$$

$$(x_1y_2 - x_2y_1 : x_2y_0 - x_0y_2 : x_0y_1 - x_1y_0)$$

$$x_0x_1y_2 - x_0x_2y_1 + x_1x_2y_0 - x_1x_0y_2 + x_2x_0y_1 - x_2x_1y_0 = 0$$

tilsvarende for $(y_0 : y_1 : y_2)$

Punkt $\longleftrightarrow$ Linje

Linje gjennom 2 punkter $\longleftrightarrow$ skjæring mellom to linjer

$\{\bar{OA}, \bar{OB}, \bar{OC}, \bar{OD}\} \longleftrightarrow \{P, Q, R, S\}$

linjer som møtes i ett punkt — punkter på linje

Dualpunkt til ℓ : L $\quad A = \ell \cap \bar{OA}$ svarer til linje gjennom L og P

$$\begin{vmatrix} x & y & z \\ L_1 & L_2 & L_3 \\ P_1 & P_2 & P_3 \end{vmatrix} = 0 \qquad \begin{pmatrix} i & j & k \\ L_1 & L_2 & L_3 \\ P_1 & P_2 & P_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

dvs linja har normalvektor $L \times P = A$

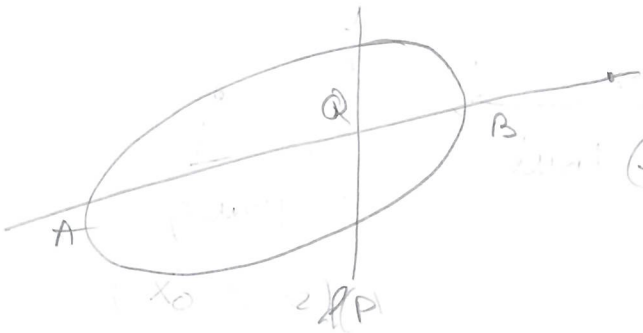
$B \rightsquigarrow L \times Q$, $C \rightsquigarrow L \times R$ $D \rightsquigarrow L \times S$

Vi har $R = pP + qQ$, $S = rP + sQ$.

$$\Rightarrow (RS, PQ) = \frac{qr}{ps}$$

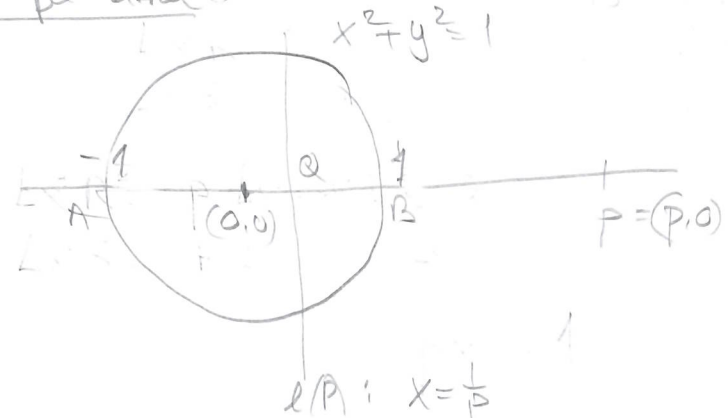
Men $L \times R = p(L \times P) + q(L \times Q)$ $\quad L \times S = r(L \times P) + s(L \times Q)$

og $(AB, CD) = \frac{qr}{ps}$.



$(A,B)(P,Q) = -1$

Barns på sirkel



$\ell(P): X = \frac{1}{p}$

$$(AB, PQ) = \frac{\overline{AP}}{\overline{AQ}} \cdot \frac{\overline{BQ}}{\overline{BP}} = \frac{p+1}{1+\frac{1}{p}} \cdot \frac{-1+\frac{1}{p}}{p-1} = \frac{(p+1)(p+1)}{(p+1)(p-1)} = -1$$

Eksempel

$X_1 = 0$ i P^2 $(X,0) = (X:0:1)$ $\infty = (1:0:0)$

$A = (0,0)$
$B = (3,0)$
$C = (1,0)$

$(AB,CD) = -1$

Setter $C = pA + qB$ $D = rA + sB$

$$\Rightarrow \frac{qr}{ps} = -1$$

$$C: (1:0:1) = p(0:0:1) + q(3:0:1)$$

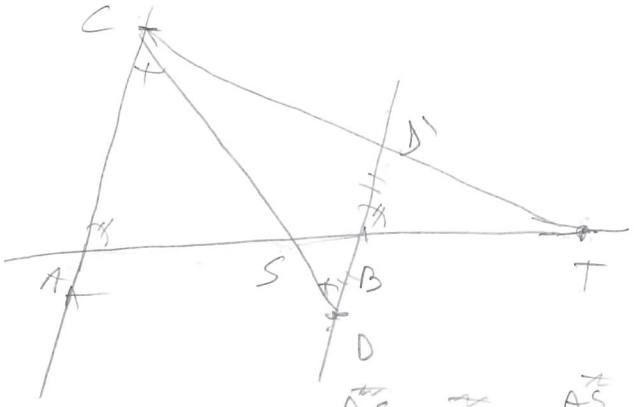
$$\Rightarrow 3q = 1,\ p+q = 1$$

$$\Rightarrow q = \frac{1}{3},\ p = \frac{2}{3}$$

dvs $\frac{\frac{1}{3}r}{\frac{2}{3}s} = -1 \Rightarrow r = -2s$

$$\Rightarrow D = -2s(0:0:1) + s(3:0:1) = (3s:0:-s)$$

$\sim (-3:0:1) \Rightarrow D = (-3,0)$



$$(AB,ST) = \frac{\overrightarrow{AS}}{\overrightarrow{AT}} \cdot \frac{\overrightarrow{BT}}{\overrightarrow{BS}} = \frac{\overrightarrow{AS}}{\overrightarrow{BS}} \cdot \frac{\overrightarrow{BT}}{\overrightarrow{AT}} = \frac{AC}{BD} \cdot \frac{BD}{AC} = 1$$

Exempel



Vi mäter i bilden: $(AB,CD) = K$.

I verkligheten $(AB,CD) = \frac{d}{d+l} \cdot \frac{d}{d-l} = \frac{d^2}{d^2-l^2} = K$

Kryssförhållandet är bevarat under proj. transf.

$$d^2 = d^2K - l^2K$$

$$\Leftrightarrow$$

$$d^2(1-K) = -l^2K$$

$$\Leftrightarrow$$

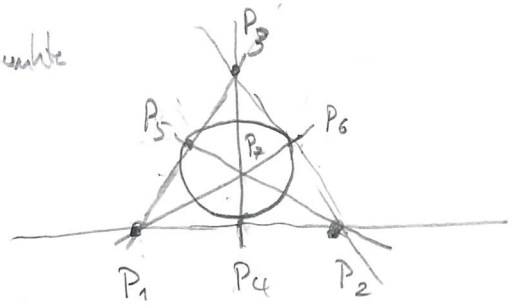
$$d^2 = \frac{K}{K-1} l^2$$

$$\Leftrightarrow$$

$$d = \sqrt{\frac{K}{K-1}}\, l$$

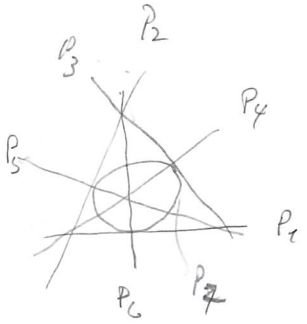
Endelig projektivt plan

Alle linjer har 3 punkter
Alle linjer møtes



7 punkter
7 linjer

P_1	P_2	P_3	P_4	P_5	P_6	P_7	
1	1	0	1	0	0	0	$P_1P_4P_2$
1	0	1	0	1	0	0	$P_1P_3P_5$
0	1	1	0	0	1	0	$P_2P_3P_6$
1	0	0	0	0	1	1	$P_1P_6P_7$
0	1	0	0	1	0	1	$P_2P_5P_7$
0	0	1	1	0	0	1	$P_3P_4P_7$
0	0	0	1	1	1	0	$P_4P_5P_6$



Symmetrisk ⇔ dualitet

Oppg 6.6.1

$P = (1:0:0)$
$Q = (1:1:0)$
$R = (1:0:1)$
$S = (1:1:1)$

Linje PQ: $\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = z = 0$

Linje RS: $\begin{vmatrix} x & y & z \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -x + z = 0$

Snitt mellom PQ og RS

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -\bar{j} \sim (0:1:0)$$

Sjekk

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 + 1 = 0.$$