Torsdag 22. oktober

Kjeglesnitt (kvadratiske former)

Repetisjon fra $\mathbb{R}^2$:

Degenererte: $\{\emptyset$, punkt, to linjer, dobbel linje$\}$

Tre typer (ikke-degenererte)

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hyperbel $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parabel $y = kx^2$

Utvidelse til $\mathbb{P}^2$:

Homogenisering gir flater i $\mathbb{R}^3$

Generell form:

$$a x_0^2 + b x_0 x_1 + c x_1^2 + d x_1 x_2 + e x_2^2 + f x_0 x_2 = 0$$

Hvis vi setter $x_2 = 1$:

$$a x_0^2 + b x_0 x_1 + c x_1^2 + d x_1 + f x_0 + e = 0$$

Har diskriminant $D = b^2 - 4ac$

Flytte origo: $d = f = 0$ (gir for parabel)

Rotasjon: $b = 0$

$\Rightarrow$ Alle de tre formene.

Karakterisert av kvadratisk form

$$(x_0\ x_1)\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \rightsquigarrow \lambda_1 x^2 + \lambda_2 y^2$$

$$\chi(\lambda) = \lambda^2 - (a+c)\lambda + ac - \frac{b^2}{4} = 0$$

Del gr $\lambda = \frac{a+c \pm \sqrt{(a+c)^2 - 4(ac - \frac{b^2}{4})}}{2}$

og $\lambda_1 \cdot \lambda_2 = \frac{1}{4}(b^2 - 4ac) = -\frac{1}{4}D$

$\Rightarrow$

$D < 0$ Egenverdier med samme fortegn: Ellipse

$D = 0$ En egenverdi er 0: Parabel

$D > 0$ Ulike fortegn for egenverdier: Hyperbel.

Eksempel

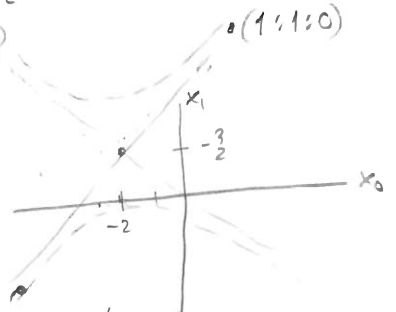
$$X_0^2 - X_1^2 + 3X_1X_2 + 4X_2^2 + 4X_0X_2 = 0$$

$X_2 = 0$: $X_0^2 - X_1^2 = 0$ To punkter $(1:1:0)$ $(1:-1:0)$

$X_1 = 0$: $X_0^2 + 4X_2^2 + 4X_0X_2 = (X_0 + 2X_2)^2 = 0$ Ett punkt $(2:0:-1)$

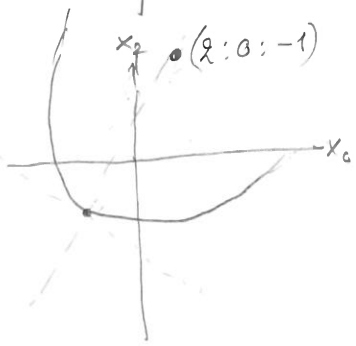
$X_0 = 0$: $-X_1^2 + 3X_1X_2 + 4X_2^2 = 0$

$$X_1 = \frac{-3 \pm \sqrt{9+16}}{-2} X_2 = \begin{cases} 4X_2 \\ -X_2 \end{cases}$$ To punkter $(0:4:1)$, $(0:-1:1)$



$X_2 = 1$: $X_0^2 - X_1^2 + 3X_1 + 4 + 4X_0 = 0$

$$(X_0+2)^2 - (X_1 - \tfrac{3}{2})^2 = -\frac{9}{4}$$



$X_1 = 1$: $X_0^2 - 1 + 3X_2 + 4X_2^2 + 4X_0X_2 = 0$

$$(X_0 + 2X_2)^2 + 3X_2 = 1$$

ella $(X_0+2X_2)^2 + \frac{6}{5}(X_0+2X_2) - \frac{3}{5}(2X_0 - X_2) = 1$

$$(X_0 + 2X_2 + \tfrac{3}{5})^2 - \tfrac{3}{5}(2X_0 - X_2 + \tfrac{3}{5}) = 1$$

Eksempel

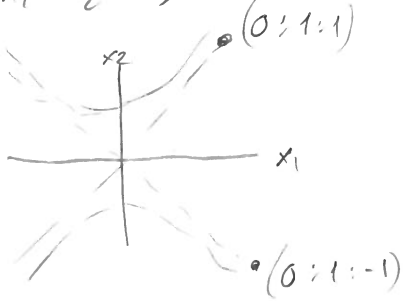
$$X_0^2 + X_1^2 - X_2^2 = 0$$

$X_0 = 0$: $X_1^2 - X_2^2 = 0$ To punkter $(0:1:1)$ $(0:1:-1)$

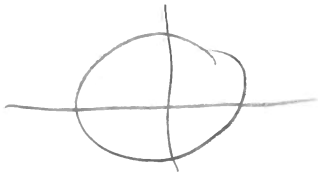
$X_2 = 0$: Ingen løsning $(X_0 = X_1 = X_2 = 0)$

$X_0 = 1$: $1 + X_1^2 - X_2^2 = 0$

$$X_2^2 - X_1^2 = 1$$



$X_2 = 1$: $X_0^2 + X_1^2 = 1$



Setning. $\exists$ projektiv transformasjon som tar ett ikke-degenerert kjeglesnitt til $x_0^2 + x_1^2 - x_2^2 = 0$

$$Q = (x_0\ x_1\ x_2)\begin{pmatrix} a & \frac{1}{2}b & \frac{1}{2}d \\ \frac{1}{2}b & c & \frac{1}{2}e \\ \frac{1}{2}d & \frac{1}{2}e & f \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_0^2$$

Symmetrisk $\Rightarrow$ Ortogonal diagonalisert gir

$$\lambda_0 x^2 + \lambda_1 y^2 + \lambda_2 z^2$$

Setter $=0$:

$\lambda_0, \lambda_1, \lambda_2 > 0$: Ingen løsning

$\lambda_0, \lambda_1 = 0$: Dobbel linje

$\lambda_0 = 0 \quad \lambda_1, \lambda_2 > 0$: Ett punkt $(1:0:0)$

$\lambda_0 = 0 \quad \lambda_1 > 0 \quad \lambda_2 < 0$: To linjer

$\lambda_0 < 0 \quad \lambda_1 \lambda_2 > 0$: Hyperbel / ellipse / parabel

Eksempel

$$x_0^2 - x_1^2 + 3x_1x_2 + 4x_2^2 + 4x_0x_2 = 0$$

$$= (x_0\ x_1\ x_2)\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & \frac{3}{2} \\ 2 & \frac{3}{2} & 4 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

3 egenverdier $\approx -1{,}562\ ,\ 0{,}272\ ,\ 5{,}290$

$\chi(\lambda) = \lambda^3 - 4\lambda^2 - \frac{29}{4}\lambda + \frac{9}{4}$

Hvordan kan $\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = 0$ bli en parabel

Eksempel $Q = y_0^2 + 4y_1^2 - y_2^2 = 0$

Basisskifte $y_0 = x_0\ ,\ y_1 = x_1 + x_2 \quad y_2 = 2x_1 + x_2$

$$\Rightarrow Q = x_0^2 + 4(x_1 + x_2)^2 - (2x_1 + x_2)^2$$

$$= x_0^2 + 4x_1x_2 + 3x_2^2$$

Setter vi $x_2 = 1$ får vi $x_0^2 + 4x_1 + 3 = 0$ (parabel)

Illustrasjon:

Halvkule sett ovenfra
- Det indre gir (x:y:1)
- Sirkelen (halve) gir (x:y:0)



Ulike projeksjoner
Samme kurve





Setning:

5 punkter i generell posisjon bestemmer et kjeglesnitt (Generell posisjon = Ingen 3 på linje)

Bevis. La $P_1 = (1:0:0)$ $P_2 = (0:1:0)$ $P_3 = (0:0:1)$

$P_4 = (1:1:1)$ $P_5 = (y_0 : y_1 : y_2)$

og $Q = ax_0^2 + bx_0x_1 + cx_1^2 + dx_1x_2 + ex_2^2 + fx_0x_2$

P_1, P_2, P_3 gir $a = c = e = 0$

P_4 gir $b + d + f = 0$

P_5 gir $by_0y_1 + dy_1y_2 + fy_0y_2 = 0$

Som vi kan skrive

$$\begin{pmatrix} 1 & 1 & 1 \\ y_0y_1 & y_1y_2 & y_0y_2 \end{pmatrix} \begin{pmatrix} b \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hvis $(y_0y_1 : y_1y_2 : y_0y_2) \neq (1:1:1)$, så har vi en entydig løsning

$(y_0y_1 : y_1y_2 : y_0y_2) = (1:1:1)$

$\Rightarrow (y_0y_1y_2 : y_0y_1y_2 : y_0y_1y_2) = (y_2 : y_0 : y_1)$

$\parallel$

$(1:1:1)$ Men $P_4 \neq P_5$ så ok

Eksplisitt: 5 punkter $(\alpha_i : \beta_i : \gamma_i)$ $i = 1,2,3,4,5$

$$Q = \det \begin{pmatrix} x_0^2 & x_0x_1 & x_1^2 & x_0x_2 & x_1x_2 & x_2^2 \\ \alpha_1^2 & \alpha_1\beta_1 & \beta_1^2 & \alpha_1\gamma_1 & \beta_1\gamma_1 & \gamma_1^2 \\ \alpha_2^2 & \alpha_2\beta_2 & \beta_2^2 & \cdots & - & \\ \alpha_3^2 & \alpha_3\beta_3 & \vdots & \ddots & & \\ \alpha_4^2 & \cdots & \cdots & & & \\ \alpha_5^2 & \vdots & & & & \gamma_5^2 \end{pmatrix}$$

Eksempel

$(1:0:0)$
$(0:1:0)$
$(0:0:1)$
$(1:1:1)$
$(y_0:y_1:y_2)$

$$Q = \det \begin{pmatrix} x_0^2 & x_0x_1 & x_1^2 & x_0x_2 & x_1x_2 & x_2^2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ y_0^2 & y_0y_1 & y_1^2 & y_0y_2 & y_1y_2 & y_2^2 \end{pmatrix}$$

$$= -\det \begin{pmatrix} x_0x_1 & x_0x_2 & x_1x_2 \\ 1 & 1 & 1 \\ y_0y_1 & y_0y_2 & y_1y_2 \end{pmatrix}$$

$$= (y_1y_2 - y_0y_2)\,x_0x_1 + (y_0y_1 - y_1y_2)\,x_0x_2 + (y_0y_2 - y_0y_1)\,x_1x_2$$

$$= y_2(y_1 - y_0)\,x_0x_1 + y_1(y_0 - y_2)\,x_0x_2 + y_0(y_2 - y_1)\,x_1x_2$$

Eksempel H20 oppg. 3 (E7, s 177)

$\mathbb{P}^2\ (x_0:x_1:x_2)$

a) $L = AB$, $A = (0:0:1)$ $B = (0:3:2)$

$L' = CD$, $C = (1:0:1)$ $D = (2:3:2)$

$$L = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \\ 0 & 3 & 2 \end{vmatrix} = -3x = 0 \quad \text{dvs } x = 0$$

$$L' = \begin{vmatrix} x & y & z \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{vmatrix} = -3x + 3z = 0 \quad \text{dvs } x - z = 0$$

$$P = L \cap L' = \{(x_0:x_1:x_2) \mid x_0 = 0,\ x_0 - x_2 = 0\} = (0:1:0)$$

b) $C_t : x_0^2 - x_0x_2 - t(2x_1^2 - 3x_1x_2) = 0$ $\quad A, B, C, D \in C_t\ \forall t$

$C_t(0:3:2) = 0^2 - 0 - t(2\cdot 3^2 - 3\cdot 3\cdot 2) = 0$

$\vdots$

Hvis $(0:1:0) \in C_t$ må vi ha $0^2 - 0\cdot 0 - t(2\cdot 1^2 - 3\cdot 1\cdot 0) = -2t = 0$

dvs $t = 0$

$C_0 : x_0^2 - x_0x_2 = x_0(x_0 - x_2) = 0$ To linjer $x_0 = 0$ $x_0 - x_2 = 0$

altså L og L'

c) $\mathbb{R}^2 \hookrightarrow \mathbb{P}^2$
$(x,y) \mapsto (x:y:1)$

$C_t(x_2 = 1) : x_0^2 - x_0 - t(2x_1^2 - 3x_1) = 0$

$= (x_0 - \frac{1}{2})^2 - 2t(x_1 - \frac{3}{4})^2 = \frac{1}{4} - \frac{9}{8}t$ Ellipse. $t < 0$

$(1,4,2) \in C_t \Rightarrow 1 - 2 - t(32 - 24) = -1 - 12t = 0 \Leftrightarrow t = -\frac{1}{12}$

$(x_0 - \frac{1}{2})^2 + \frac{1}{6}(x_1 - \frac{3}{4})^2 = \frac{1}{4} + \frac{9}{8\cdot 12} = \frac{11}{32}$