UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT2700 — Introduction to mathematical finance and investment theory.

Day of examination: Monday, December 14, 2015.

Examination hours: 09:00 – 13:00.

This problem set consists of 5 pages.

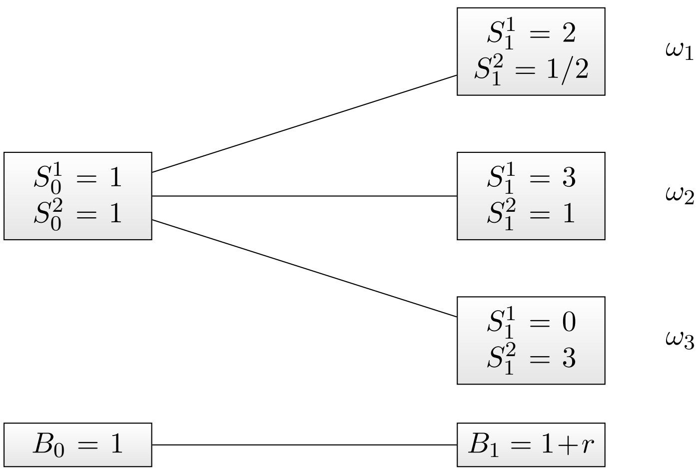
Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider the following model with three scenarios and two stocks and one bond with interest rate r.



1a

Show that the model is not viable if $r = 0$, and find an arbitrage opportunity in this case.

Possible solution: An implicit arbitrage opportunity is found by solving $y \cdot \Delta S(\omega_i) \geq 0$ for $i = 1, 2, 3$. These equations are

$$\begin{aligned} y_1 - \frac{1}{2}y_2 &\geq 0, \\ 2y_1 &\geq 0, \\ -y_1 + 2y_2 &\geq 0. \end{aligned}$$

(Continued on page 2.)

One solution is $z = y_1 = y_2 > 0$, which gives the arbitrage opportunity $\varphi = (-2z, z, z)$ with

$$V_1^{\varphi}(\omega_1) = z/2, \quad V_1^{\varphi}(\omega_2) = 2z, \quad V_1^{\varphi}(\omega_3) = z.$$

Since we have arbitrage opportunities, the model is not viable.

1b

Show that the model is viable and complete for $r_0 < r < r_1$, and find r_0 and r_1.

Possible solution: The model is viable and complete if we have a unique risk free probability measure $\mathbb{Q} = (q_1, q_2, q_3)$, the equations for $\mathbb{Q}$ are

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ \frac{1}{2} & 1 & 3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+r \\ 1+r \end{pmatrix}.$$

Solving this equation, we get

$$q_1 = \frac{8 - 10r}{7}, \quad q_2 = \frac{9r - 3}{7}, \quad q_3 = \frac{r+2}{7}.$$

For this to be a probability measure, we must have $0 < q_i < 1$ for $i = 1, 2, 3$. For $i = 1$, this gives $1/10 < r < 4/5$, for $i = 2$, $1/3 < r < 10/9$, and for $i = 3$, $-2 < r < 5$. Therefore the model is viable and complete if

$$r_0 = \frac{1}{3} < r < \frac{4}{5} = r_1.$$

1c

In the rest of this exercise we assume that $r = 5/8$, then the model is viable and complete (you do not have to show this). Find the fair price of the "indicator" derivatives D^j, with payoff at $t = 1$

$$D^j(\omega_i) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \qquad \text{for } j = 1, 2, 3.$$

Possible solution: If $r = 5/8$, then the unique risk neutral probability measure is

$$\mathbb{Q} = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right).$$

The fair price of D^j is $\mathbb{E}_{\mathbb{Q}}(\bar{D}^j) = q_j/(1+r)$, i.e.,

$$\mathbb{E}_{\mathbb{Q}}(\bar{D}^1) = \frac{1}{4}\frac{8}{13} = \frac{2}{13}, \quad \mathbb{E}_{\mathbb{Q}}(\bar{D}^2) = \frac{3}{8}\frac{8}{13} = \frac{3}{13}, \quad \mathbb{E}_{\mathbb{Q}}(\bar{D}^3) = \frac{3}{8}\frac{8}{13} = \frac{3}{13}.$$

(Continued on page 3.)

1d

Suppose a third stock was valued at $t = 1$ as follows

$$S_1^3(\omega_1) = 1, \quad S_1^3(\omega_2) = 2, \quad S_1^3(\omega_3) = 3.$$

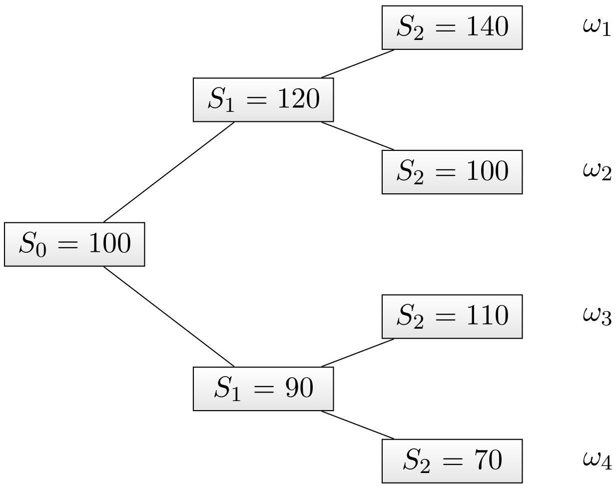
What must its price be at $t = 0$ for the model with three stocks to be viable?

Possible solution: We can view S^3 as a derivative, and the unique fair price is the one which excludes arbitrage, hence

$$S_0^3 = \mathbb{E}_{\mathbb{Q}}(\bar{S}_1^3) = 1\frac{2}{13} + 2\frac{3}{13} + 3\frac{3}{13} = \frac{17}{13}$$

Problem 2

Consider the following binary two period model with one stock and one bond.



Assume that the interest rate is 0 and that the initial bond price is $B_0 = 1$.

2a

Find the unique equivalent martingale measure $\mathbb{Q}$.

Possible solution: The risk neutral conditional probabilities are

$$\begin{aligned} & q_{uu} = 1/2 \\ q_u = 1/3 \quad & q_{ud} = 1/2 \\ q_d = 2/3 \quad & q_{du} = 1/2 \\ & q_{dd} = 1/2 \end{aligned}$$

From this we get the equivalent martingale measure

$$\mathbb{Q}(\omega_1) = \frac{1}{6} \qquad \mathbb{Q}(\omega_2) = \frac{1}{6} \qquad \mathbb{Q}(\omega_3) = \frac{1}{3} \qquad \mathbb{Q}(\omega_4) = \frac{1}{3}.$$

The model is viable and complete since we have a unique equivalent martingale measure.

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2b

Let X be the random variable defined by

$$X(\omega_1) = 0, \quad X(\omega_2) = 1, \quad X(\omega_3) = -1, \quad X(\omega_4) = 0.$$

and let $\mathcal{F}_t$ be the filtration defined by the stock prices. List $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$. For which t is X measurable with respect to $\mathcal{F}_t$?

Possible solution: Set $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\Omega_u = \{\omega_1, \omega_2\}$, $\Omega_d = \{\omega_3, \omega_4\}$, then

$$\begin{aligned} \mathcal{F}_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}_1 &= \{\emptyset, \Omega, \Omega_u, \Omega_d\}, \\ \mathcal{F}_2 &= \text{all subsets of } \Omega. \end{aligned}$$

X is measurable with respect to $\mathcal{F}_t$ iff it takes a single value on each t-node. This is the case only for $t = 2$.

2c

Find the conditional expectation $Y_t = \mathbb{E}_{\mathbb{Q}} \left\{ X \mid \mathcal{F}_t \right\}$ for $t = 0, 1$.

Possible solution: We have that

$$Y_0 = \mathbb{E}_{\mathbb{Q}}(X) = \mathbb{Q}(\omega_2) - \mathbb{Q}(\omega_3) = \frac{1}{6} - \frac{1}{3} = -\frac{1}{6}.$$

also

$$\begin{aligned} Y_1(u) &= \frac{1}{\frac{1}{3}} \mathbb{Q}(\omega_2) = \frac{1}{2}, \\ Y_1(d) &= \frac{1}{\frac{2}{3}} (-\mathbb{Q}(\omega_3)) = -\frac{1}{2}. \end{aligned}$$

2d

Find the fair price of the European call option with payoff $\max\{S_2 - 100, 0\}$.

Possible solution: We find the payoff of the derivative for each scenario,

$$D(\omega_1) = 40, \quad D(\omega_2) = 0, \quad D(\omega_3) = 10, \quad D(\omega_4) = 0.$$

Hence the fair price is

$$D_0 = \mathbb{E}_{\mathbb{Q}}(D) = \left(40\frac{1}{6} + 10\frac{1}{3} \right) = 10.$$

(Continued on page 5.)

2e

Assume that the real world probability of ω_i is

$$\mathbb{P}(\omega_1) = 1/6, \quad \mathbb{P}(\omega_2) = 1/3, \quad \mathbb{P}(\omega_3) = 1/3, \quad \mathbb{P}(\omega_4) = 1/6.$$

Given an initial wealth V_0, find a self financing trading strategy $\Phi = \{(x_t, y_t)\}_{t=1}^2$ which maximizes

$$\mathbb{E}_{\mathbb{P}}\left[u(V_2^\Phi)\right], \quad \text{given that } V_0^\Phi = V_0,$$

for the utility function $u(v) = \ln(v)$.

Possible solution: We use the martingale method to find this trading strategy. Since the utility function is the logarithm, the "optimal derivative" is

$$D(\omega_i) = \frac{\mathbb{P}(\omega_i)}{\mathbb{Q}(\omega_i)} V_0 = V_0 \begin{cases} 1, & i = 1, \\ 2, & i = 2, \\ 1, & i = 3, \\ 1/2, & i = 4. \end{cases}$$

We must find a portfolio for each of the two submodels at u, d. At u we must solve the equations

$$\begin{aligned} x + 140y &= V_0 \\ x + 100y &= 2V_0. \end{aligned}$$

This gives

$$x_2(u) = (9/2)V_0, \quad y_2(u) = -(1/40)V_0, \quad V_2^\Phi(u) = (3/2)V_0.$$

Similarly for the node d

$$x_2(d) = -(3/8)V_0, \quad y_2(d) = (1/80)V_0, \quad V_2^\Phi(d) = (3/4)V_0.$$

The equations for the root node are

$$\begin{aligned} x + 120y &= (3/2)V_0 \\ x + 90y &= (3/4)V_0, \end{aligned}$$

giving

$$x_1 = -(3/2)V_0, \quad y_1 = (1/40)V_0, \quad V_1^\Phi = V_0.$$

THE END