UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: Friday, December 2nd, 2016

Examination hours: 09.00 – 13.00

This problem set consists of 9 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider a single-period market consisting of a bank account B and one risky asset S_1. The bank is given by $B(0) = 1$ and $B(1) = 1+r$, where $r > 0$ is the interest rate. The sample space is $\Omega = \{\omega_1, \omega_2\}$, where the probability of ω_1 occurring is $p \in (0, 1)$. The risky asset is given by $S_1(0) = 1$, $S_1(1, \omega_1) = u$, and $S_1(1, \omega_2) = d$, where $u > d$ are two given numbers.

1a

What is the definition of a risk-neutral probability measure Q? Use this definition to show that a probability Q is risk-neutral if and only if

$$E_Q[R_1] = r, \tag{1}$$

where R_1 is the return of the risky asset S_1.

Answer: A vector $Q = (Q_1, Q_2)$, with $Q_1, Q_2 > 0$ and $Q_1 + Q_2 = 1$, is risk-neutral provided

$$E_Q[\Delta S_1^*] = 0, \quad \Delta S_1^* = S_1^*(1) - S_1^*(0) = \frac{S_1(1)}{1+r} - 1,$$

or, equivalently,

$$E_Q[S_1(1)] = 1 + r.$$

From this it follows that

$$E_Q\left[\frac{S_1(1) - S_1(0)}{S_1(0)}\right] = r,$$

that is, $E_Q[R_1] = r$.

(Continued on page 2.)

1b

Use (1) to determine the risk-neutral probability Q.

Answer: Assuming $Q = (q, 1-q)$, with $0 < q < 1$, and noting that

$$R_1 = \frac{S_1(1) - S_1(0)}{S_1(0)} = (u-1, d-1)\,,$$

it follows from (1) that

$$r = E_Q[R_1] = Q \cdot R_1 = q(u-1) + (1-q)(d-1) = q(u-d) + d - 1$$

and therefore

$$q = \frac{(1+r)-d}{u-d}, \qquad 1-q = \frac{u-(1+r)}{u-d}.$$

To ensure $0 < q < 1$ we must have

$$r - d + 1 > 0 \iff d < 1+r, \qquad r - d + 1 < u - d \iff u > 1+r.$$

1c

Specify $d = \frac{1}{2}(1+r)$ and $u = 2(1+r)$. Denote by X the payoff of a call option with exercise price $e = 1+r$. Why is X is attainable? Use 1b) to compute the price of the call option.

Answer: In view 1b), since $d < 1+r$ and $u > 1+r$, the market is complete and thus all claims are attainable. We have

$$S_1(1) - e = \begin{cases} 2(1+r) - (1+r) = 1+r, & \text{if } \omega = \omega_1 \\ \frac{1}{2}(1+r) - (1+r) = -\frac{1}{2}(1+r), & \text{if } \omega = \omega_2 \end{cases}$$

and so the payoff of the call option is

$$X = \max\left(S_1(1) - e), 0\right) = \begin{cases} 1+r, & \text{if } \omega = \omega_1 \\ 0, & \text{if } \omega = \omega_2 \end{cases}.$$

The price is

$$E_Q\left[\frac{X}{B(1)}\right] = q = \frac{(1+r)-d}{u-d} = \frac{\frac{1}{2}(1+r)}{\frac{3}{2}(1+r)} = \frac{1}{3}.$$

1d

Prove the relation

$$\overline{R_1} - r = -\mathrm{cov}(R_1, L), \tag{2}$$

where $\overline{R_1}$ is the mean return, L is the state price density, and $\mathrm{cov}(X,Y)$ is the covariation between two random variables X and Y.

Answer: The state price density is given by

$$L = \frac{Q}{P}, \qquad E[L] = 1,$$

(Continued on page 3.)

and so

$$\begin{aligned}\operatorname{cov}(R_1, L) &= E[R_1 L] - E[R_1]E[L] \\ &= E_Q[R_1] - E[R_1] \\ &= E_Q[R_1] - \overline{R_1} \\ &= r - \overline{R_1} \quad \text{by (1);}\end{aligned}$$

thus, (2) follows.

Problem 2

We continue to examine the one-period model from Problem 1, this time assuming that $r = 1$, $d = 1$, and $u = 4$.

2a

Explain why $U(w) = 2\sqrt{w}$, for $w > 0$, is a utility function. Compute the Arrow-Pratt coefficient $\alpha_R(w) = -w\frac{U''(w)}{U'(w)}$ of relative risk aversion.

<u>Answer:</u> The function U is a utility function since it is continuously differentiable, concave, and strictly increasing. Indeed, we compute easily $U'(w) = \frac{1}{w^{1/2}} > 0$ and $U''(w) = -\frac{1}{2w^{3/2}} < 0$. The Arrow-Pratt coefficient is

$$\alpha_R(w) = -w\frac{U''(w)}{U'(w)} = w\frac{\frac{1}{2w^{3/2}}}{\frac{1}{w^{1/2}}} = \frac{1}{2},$$

hence $U(w) = 2\sqrt{w}$ displays constant relative risk aversion.

2b

Consider the portfolio problem

$$\max_{H \in \mathbf{R}^2} E\left[2\sqrt{V(1)}\right], \quad V(0) = \nu, \tag{3}$$

where $V(t)$ is the portfolio value at time $t = 0, 1$ corresponding to a trading strategy $H = (H_0, H_1)$, and $\nu > 0$ is the given initial wealth. Suppose there is a solution H_{opt} to (3). Then use the optimality of H_{opt} to show that an arbitrage opportunity $\hat{H}$ cannot exist.

<u>Answer:</u> To reach a contradiction, suppose there is an arbitrage opportunity $\hat{H}$; i.e., $\hat{V}(0) = 0$, $\hat{V}(1) \geq 0$, $\hat{V}(1, \omega) > 0$ for at least one $\omega \in \Omega$, say ω_1. Set

$$H := H_{\text{opt}} + \hat{H}.$$

Then, clearly,

$$V(1) = V_{\text{opt}}(1) + \hat{V}(1) \geq V_{\text{opt}}(1) \quad \text{on } \Omega,$$

and

$$V(1, \omega_1) > V_{\text{opt}}(1, \omega_1).$$

(Continued on page 4.)

Since $U(w) = 2\sqrt{w}$ is strictly increasing,

$$\begin{aligned} E\left[U(V(1))\right] &= pU(V(1,\omega_1)) + (1-p)U(V(1,\omega_2)) \\ &> pU(V_{\text{opt}}(1,\omega_1)) + (1-p)U(V_{\text{opt}}(1,\omega_2)) \\ &= E\left[U(V_{\text{opt}}(1))\right], \end{aligned}$$

which contradicts the optimality of H_{opt}.

2c

Make use of the risk-neutral probability approach to solve the $Q = (\frac{1}{3}, \frac{2}{3})$ is the unique risk-neutral probability measure in the market. Divide your answer into two parts: 1) Determine the optimal wealth. 2) Determine the optimal trading strategy.

<u>Answer:</u> The first step is to maximize expected utility of wealth:

$$\max_{W \in \mathbb{R}^2} E[U(W)], \quad \text{subject to } E_Q\left[\frac{W}{B(1)}\right] = \nu,$$

where $U(w) = 2\sqrt{w}$ and $B(1) = 2$. We employ the Lagrange multiplier method:

$$\max_{W} \left\{ E[U[W]] - \lambda E_Q\left[\frac{W}{2}\right] \right\},$$

where the Lagrange multiplier $\lambda > 0$ is found by demanding $E_Q\left[\frac{W}{2}\right] = \nu$. Using the state price density $L = Q/P$ we can write

$$E[U(W)] - \lambda E_Q\left[\frac{W}{2}\right] = E\left[U(W) - \lambda \frac{LW}{2}\right].$$

The first order condition at a maximum reads:

$$U'(W) = \lambda \frac{L}{2}.$$

Denote the inverse of $U'(w) = \frac{1}{\sqrt{w}}$ by I, so

$$I(y) = \frac{1}{y^2}.$$

It then follows that the optimal wealth $\hat{W}$ is

$$W_{\text{opt}} = I\left(\lambda \frac{L}{2}\right) = \frac{4}{\lambda^2 L^2}.$$

We identify $\lambda > 0$ using $E_Q\left[\frac{W_{\text{opt}}}{2}\right] = \nu$:

$$\begin{aligned} E_Q\left[\frac{2}{\lambda^2 L^2}\right] = \nu &\iff \frac{1}{\lambda^2} E_Q\left[\frac{1}{L^2}\right] = \frac{\nu}{2} \\ &\iff \lambda = \left(\frac{2E_Q\left[\left(\frac{P}{Q}\right)^2\right]}{\nu}\right)^{\frac{1}{2}}. \end{aligned}$$

(Continued on page 5.)

Since

$$E_Q\left[\left(\frac{P}{Q}\right)^2\right] = \frac{1}{3}\left(\frac{p}{\frac{1}{3}}\right)^2 + \frac{2}{3}\left(\frac{1-p}{\frac{2}{3}}\right)^2 = 3p^2 + \frac{3}{2}(1-p)^2,$$

we obtain $\lambda = \sqrt{\frac{\kappa}{\nu}}$, with $\kappa = 6p^2 + 3(1-p)^2 = 3(3p^2 - 2p + 1)$. This gives the optimal wealth W_{opt}:

$$W_{\text{opt}} = \frac{4\nu}{\kappa}\left(\frac{P}{Q}\right)^2 = \begin{cases} \frac{36\nu p^2}{\kappa}, & \omega = \omega_1 \\ \frac{9\nu(1-p)^2}{\kappa}, & \omega = \omega_2 \end{cases}.$$

The second step is to find the optimal trading strategy H_{opt}. We seek a vector $H = (H_0, H_1)$ such that $V(1) = H_0 B(1) + H_1 S_1(1) = W_{\text{opt}}$, that is,

$$\begin{aligned} 2H_0 + 4H_1 &= \frac{36\nu p^2}{\kappa}, \\ 2H_0 + H_1 &= \frac{9\nu(1-p)^2}{\kappa}. \end{aligned}$$

The solution to this system is $H_0 = \frac{6(1-2p)\nu}{\kappa}$ and $H_1 = \frac{3(3p^2+2p-1)\nu}{\kappa}$, and so

$$H_{\text{opt}} = \left(\frac{6(1-2p)\nu}{\kappa}, \frac{3\left(3p^2+2p-1\right)\nu}{\kappa}\right).$$

Problem 3

Consider a multi-period market ($T = 2$) with one risky asset evolving as follows:

$$S_1(0) = 1, \quad S_1(1,\omega) = \begin{cases} \frac{3}{2}, & \omega = \omega_1, \omega_2 \\ \frac{1}{2}, & \omega = \omega_3, \omega_4 \end{cases}, \quad S_1(2,\omega) = \begin{cases} \frac{9}{4}, & \omega = \omega_1 \\ \frac{3}{4}, & \omega = \omega_2 \\ \frac{3}{4}, & \omega = \omega_3 \\ \frac{1}{8}, & \omega = \omega_4 \end{cases}.$$

The bank pays zero interest, i.e., $B(0) = 1$, $B(1) = 1$, and $B(2) = 1$. Moreover, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and the probability measure is

$$P(\omega) = \begin{cases} 1/4, & \omega = \omega_1 \\ 1/4, & \omega = \omega_2 \\ 1/4, & \omega = \omega_3 \\ 1/4, & \omega = \omega_4 \end{cases}.$$

3a

Identify the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,1,2}$ generated by the risky asset. Compute the conditional expectation

$$E\left[S_1(2) \middle| \mathcal{F}_1\right].$$

Verify that

$$E\left[E\left[S_1(2) \middle| \mathcal{F}_1\right]\right] = E\left[S_1(2)\right]. \tag{4}$$

(Continued on page 6.)

Answer: We read off the filtration from the tree in Figure 1:

$$\mathcal{P}_0 = \{\Omega\}, \quad \mathcal{F}_0 = \{\Omega, \emptyset\},$$

$$\mathcal{P}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \quad \mathcal{F}_1 = \{\Omega, \emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\},$$

$$\mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}, \quad \mathcal{F}_2 = \text{the collection of all subsets of } \Omega.$$

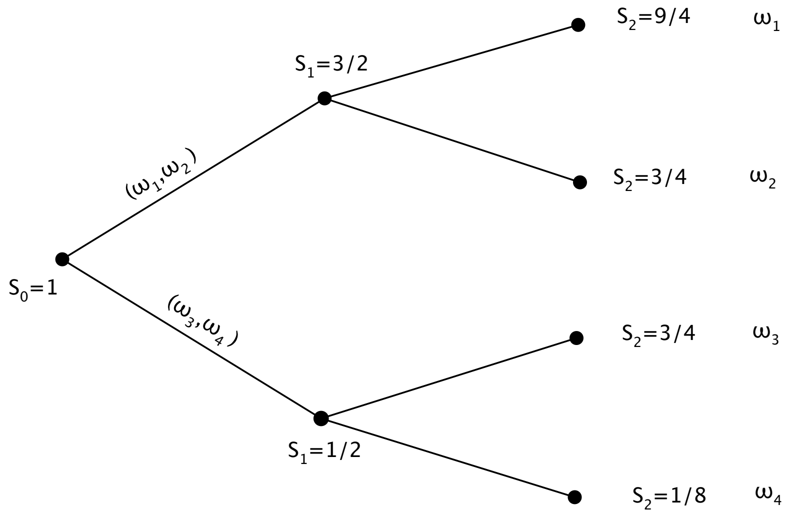

Figure 1: Filtration generated by the price process (Problem 3).

By definition,

$$\begin{aligned}
E\left[S_1(2)\middle|\mathcal{F}_1\right] &= \sum_{A\in\mathcal{P}_1} E\left[S_1(2)\middle|A\right] \mathbf{1}_A(\omega) \\
&= \begin{cases} E\left[S_1(2)\middle|A_1\right], & \omega \in A_1 := \{\omega_1, \omega_2\} \\ E\left[S_1(2)\middle|A_2\right], & \omega \in A_2 := \{\omega_3, \omega_4\} \end{cases} \\
&= \begin{cases} S_1(2,\omega_1)\frac{P(\omega_1)}{P(A_1)} + S_1(2,\omega_2)\frac{P(\omega_2)}{P(A_1)}, & \omega \in A_1 \\ S_1(2,\omega_3)\frac{P(\omega_3)}{P(A_2)} + S_1(2,\omega_4)\frac{P(\omega_4)}{P(A_2)}, & \omega \in A_2 \end{cases} \\
&= \begin{cases} \frac{9}{4}\cdot\frac{\frac{1}{4}}{\frac{1}{2}} + \frac{3}{4}\cdot\frac{\frac{1}{4}}{\frac{1}{2}}, & \omega \in A_1 \\ \frac{3}{4}\cdot\frac{\frac{1}{4}}{\frac{1}{2}} + \frac{1}{8}\cdot\frac{\frac{1}{4}}{\frac{1}{2}}, & \omega \in A_2 \end{cases} \\
&= \begin{cases} \frac{3}{2}, & \omega \in A_1 \\ \frac{7}{16}, & \omega \in A_2 \end{cases}.
\end{aligned}$$

Since

$$E\left[E\left[S_1(2)\middle|\mathcal{F}_1\right]\right] = \left(\frac{3}{2}\cdot\frac{1}{4} + \frac{3}{2}\cdot\frac{1}{4}\right) + \left(\frac{7}{16}\cdot\frac{1}{4} + \frac{7}{16}\cdot\frac{1}{4}\right) = \frac{31}{32}$$

and

$$E\left[S_1(2)\right] = \frac{9}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \frac{1}{8}\cdot\frac{1}{4} = \frac{31}{32},$$

the claim (4) follows.

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3b

Determine the risk-neutral probability (martingale) measure Q.

Answer: We determine $Q = (Q_1, Q_2, Q_3, Q_4)$ using

$$E_Q\left[S_1^*(t+s)\middle|\mathcal{F}_t\right] = S_1^*(t), \qquad t = 0,1,\ s = 1,2,$$

that is,

$$E_Q\left[S_1(t+s)\middle|\mathcal{F}_t\right] = S_1(t), \qquad t = 0,1,\ s = 1,2.$$

Moreover,

$$Q_1 + Q_2 + Q_3 + Q_4 = 1. \tag{5}$$

$t = 0$: With $s = 1$, the condition is

$$E_Q\left[S_1(1)\middle|\mathcal{F}_0\right] = S_1(0),$$

which reads

$$\frac{3}{2}(Q_1 + Q_2) + \frac{1}{2}(Q_3 + Q_4) = 1. \tag{6}$$

With $s = 2$, the condition is

$$E_Q\left[S_1(2)\middle|\mathcal{F}_0\right] = S_1(0),$$

which reads

$$\frac{9}{4}Q_1 + \frac{3}{4}(Q_2 + Q_3) + \frac{1}{8}Q_4 = 1. \tag{7}$$

$t = 1$: With $s = 1$, the condition is

$$E_Q\left[S_1(2)\middle|\mathcal{F}_1\right] = S_1(1).$$

By definition,

$$\begin{aligned}
E_Q\left[S_1(2)\middle|\mathcal{F}_1\right] &= \sum_{A\in\mathcal{P}_1} E_Q\left[S_1(2)\middle|A\right] \mathbf{1}_A(\omega) \\
&= \begin{cases} E_Q\left[S_1(2)\middle|A_1\right], & \omega \in A_1 := \{\omega_1, \omega_2\} \\ E_Q\left[S_1(2)\middle|A_2\right], & \omega \in A_2 := \{\omega_3, \omega_4\} \end{cases}.
\end{aligned}$$

We continue by computing

$$\begin{aligned}
E_Q\left[S_1(2)\middle|A_1\right] &= \sum_{\omega\in A_1} S_1(2,\omega)\frac{Q(\omega)}{Q(A_1)} \\
&= S_1(2,\omega_1)\frac{Q(\omega_1)}{Q(A_1)} + S_1(2,\omega_2)\frac{Q(\omega_2)}{Q(A_1)} \\
&= \left(\frac{9}{4}Q_1 + \frac{3}{4}Q_2\right)/(Q_1 + Q_2)
\end{aligned}$$

and

$$\begin{aligned}
E_Q\left[S_1(2)\middle|A_2\right] &= \sum_{\omega\in A_2} S_1(2,\omega)\frac{Q(\omega)}{Q(A_2)} \\
&= S_1(2,\omega_3)\frac{Q(\omega_3)}{Q(A_2)} + S_1(2,\omega_4)\frac{Q(\omega_4)}{Q(A_2)} \\
&= \left(\frac{3}{4}Q_3 + \frac{1}{8}Q_4\right)/(Q_3 + Q_4),
\end{aligned}$$

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and obtain therefore the equations

$$\left(\frac{9}{4}Q_1 + \frac{3}{4}Q_2\right)/(Q_1 + Q_2) = S_1(\omega) = \frac{3}{2},$$

for $\omega = \omega_1, \omega_2$, and

$$\left(\frac{3}{4}Q_3 + \frac{1}{8}Q_4\right)/(Q_3 + Q_4) = S_1(\omega) = \frac{1}{2},$$

for $\omega = \omega_3, \omega_4$. Slightly rewriting, we arrive finally at

$$9Q_1 + 3Q_2 = 6(Q_1 + Q_2) \Longleftrightarrow Q_1 - Q_2 = 0 \tag{8}$$

and

$$6Q_3 + Q_4 = 4(Q_3 + Q_4) \Longleftrightarrow 2Q_3 - 3Q_4 = 0. \tag{9}$$

Solving (5)–(9) gives

$$Q_1 = \frac{1}{4}, \quad Q_2 = \frac{1}{4}, \quad Q_3 = \frac{3}{10}, \quad Q_4 = \frac{1}{5}.$$

3c

Denote by X the payoff of a put option with exercise price $e = 1$. The option expires at $T = 2$. What does it mean for X to be attainable (marketable)? Why is X attainable? Use the risk-neutral pricing formula to compute the $t = 1$ value of X.

Answer: The claim X is attainable if there exists a self-financing trading strategy such that $V_2 = X$. Since there is a unique martingale measure $Q = \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{10}, \frac{1}{5}\right)$, the market is complete and thus all claims are attainable.

Since

$$e - S_1(2) = \begin{cases} 1 - \frac{9}{4} = -\frac{5}{4}, & \omega = \omega_1 \\ 1 - \frac{3}{4} = \frac{1}{4}, & \omega = \omega_2 \\ 1 - \frac{3}{4} = \frac{1}{4}, & \omega = \omega_3 \\ 1 - \frac{1}{8} = \frac{7}{8}, & \omega = \omega_4 \end{cases},$$

the payoff is

$$X = \max\left(e - S_1(2), 0\right) = \begin{cases} 0, & \omega = \omega_1 \\ \frac{1}{4}, & \omega = \omega_2 \\ \frac{1}{4}, & \omega = \omega_3 \\ \frac{7}{8}, & \omega = \omega_4 \end{cases}.$$

The $t = 1$ value of X is

$$E_Q\left[\frac{X}{B(2)}\Big|\mathcal{F}_1\right] = E_Q\left[X\big|\mathcal{F}_1\right] \quad \text{(since } (B(2) = 1).$$

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We compute the conditional expectation $E_Q\left[X\middle|\mathcal{F}_1\right]$ as before:

$$
\begin{aligned}
E_Q\left[X\middle|\mathcal{F}_1\right] &= \sum_{A\in\mathcal{P}_1} E_Q\left[X\middle|A\right]\mathbf{1}_A(\omega) \\
&= \begin{cases} E_Q\left[X\middle|A_1\right], & \omega\in A_1 := \{\omega_1,\omega_2\} \\ E_Q\left[X\middle|A_2\right], & \omega\in A_2 := \{\omega_3,\omega_4\} \end{cases} \\
&= \begin{cases} X(\omega_1)\frac{Q(\omega_1)}{Q(A_1)} + X(\omega_2)\frac{Q(\omega_2)}{Q(A_1)}, & \omega\in A_1 \\ X(\omega_3)\frac{Q(\omega_3)}{Q(A_2)} + X(\omega_4)\frac{Q(\omega_4)}{Q(A_2)}, & \omega\in A_2 \end{cases} \\
&= \begin{cases} 0\cdot\frac{\frac{1}{4}}{\frac{1}{2}} + \frac{1}{4}\cdot\frac{\frac{1}{4}}{\frac{1}{2}}, & \omega\in A_1 \\ \frac{1}{4}\cdot\frac{\frac{3}{10}}{\frac{1}{2}} + \frac{7}{8}\cdot\frac{\frac{1}{5}}{\frac{1}{2}}, & \omega\in A_2 \end{cases} \\
&= \begin{cases} \frac{1}{8}, & \omega\in A_1 \\ \frac{1}{2}, & \omega\in A_2 \end{cases}.
\end{aligned}
$$

Hence, the $t=1$ value of X is $\begin{cases} \frac{1}{8}, & \omega\in\{\omega_1,\omega_2\} \\ \frac{1}{2}, & \omega\in\{\omega_3,\omega_4\} \end{cases}$.

THE END