UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3100 — Linear optimization.

Day of examination: Wednesday 11. June 2025.

Examination hours: 09:00 – 13:00.

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 (Simplex method). We will consider the following linear programming problem.

$$\begin{array}{lrcrcrcl} \max & 4x_1 & + & 2x_2 & + & x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 1 \\ & \frac{1}{2}x_1 & - & x_2 & & & \leq & 0 \\ & -2x_1 & + & x_2 & & & \leq & 0 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array} \tag{1}$$

a) Write down the dictionary corresponding to (1), and solve the problem using the simplex method and the largest coefficient rule. Write also down the optimal value and optimal solution. Is the optimal solution unique?
Solution: The dictionary is

$$\begin{array}{lcrcrcrcr} \eta & = & & & 4x_1 & + & 2x_2 & + & x_3 \\ w_1 & = & 1 & - & x_1 & - & x_2 & - & x_3 \\ w_2 & = & & & -\frac{1}{2}x_1 & + & x_2 & & \\ w_3 & = & & & 2x_1 & - & x_2 & & \end{array}$$

According to the largest coefficient rule, x_1 is the entering variable. The ratios are 1, ∞, and $-\infty$, respectively. The biggest is ∞, so that w_2 is the leaving variable. Substituting $x_1 = -2w_2 + 2x_2$ we get

$$\begin{array}{lcrcrcrcr} \eta & = & & - & 8w_2 & + & 10x_2 & + & x_3 \\ w_1 & = & 1 & + & 2w_2 & - & 3x_2 & - & x_3 \\ x_1 & = & & - & 2w_2 & + & 2x_2 & & \\ w_3 & = & & - & 4w_2 & + & 3x_2 & & \end{array}$$

x_2 is now the entering variable. The ratios are now 3, $-\infty$, and $-\infty$, respectively. 3 is the biggest, so that w_1 is the leaving variable. Substituting

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$x_2 = \frac{1}{3} + \frac{2}{3}w_2 - \frac{1}{3}w_1 - \frac{1}{3}x_3$ we get

$$\begin{array}{rcrcrcrcr} \eta & = & \frac{10}{3} & - & \frac{4}{3}w_2 & - & \frac{10}{3}w_1 & - & \frac{7}{3}x_3 \\ x_2 & = & \frac{1}{3} & + & \frac{2}{3}w_2 & - & \frac{1}{3}w_1 & - & \frac{1}{3}x_3 \\ x_1 & = & \frac{2}{3} & - & \frac{2}{3}w_2 & - & \frac{2}{3}w_1 & - & \frac{2}{3}x_3 \\ w_3 & = & 1 & - & 2w_2 & - & w_1 & - & x_3 \end{array}$$

This dictionary is optimal, with optimal value $10/3$, and $(2/3, 1/3, 0)$ as the unique optimizer. The slack variables are $w_1 = w_2 = 0$, and $w_3 = 1$.

b) What does it mean for a dictionary to be degenerate, and for a pivot to be degenerate? Which dictionaries/pivots in **a)** were degenerate?
Solution: A dictionary is degenerate if at least one the constants in the constraints is zero. A pivot is degenerate if it does not increase the objective value. We see that the original and the next dictionary are degenerate. The first pivot is degenerate.

c) Write down the dual problem and the optimal dual dictionary. What is the optimal solution to the dual problem? Is this solution unique?
Solution: The dual problem is

$$\begin{array}{lrcrcrcr} \min & y_1 & & & & & & \\ \text{subject to} & y_1 & + & \frac{1}{2}y_2 & - & 2y_3 & \geq & 4 \\ & y_1 & - & y_2 & + & y_3 & \geq & 2 \\ & y_1 & & & & & \geq & 1 \\ & & & & & y_1, y_2, y_3 & \geq & 0 \end{array}$$

The optimal dual dictionary is the negative transpose of the optimal primal dictionary:

$$\begin{array}{rcrcrcrcr} -\zeta & = & -\frac{10}{3} & - & \frac{1}{3}z_2 & - & \frac{2}{3}z_1 & - & y_3 \\ y_2 & = & \frac{4}{3} & - & \frac{2}{3}z_2 & + & \frac{2}{3}z_1 & + & 2y_3 \\ y_1 & = & \frac{10}{3} & + & \frac{1}{3}z_2 & + & \frac{2}{3}z_1 & + & y_3 \\ z_3 & = & \frac{7}{3} & + & \frac{1}{3}z_2 & + & \frac{2}{3}z_1 & + & y_3 \end{array}$$

Here variables have been replaced with their complementary counterparts. We see that the optimal objective value is $\zeta = \frac{10}{3}$, and we obtain the optimal solution $y = (10/3, 4/3, 0)$, which is also unique.

d) What is the optimal solution if the objective function in (1) is changed to $6x_1 + 3x_2 + 5x_3$? Is the optimal solution unique now?
Solution: Using the optimal dictionary from **a)** we rewrite the objective function as

$$\begin{aligned} &6x_1 + 3x_2 + 5x_3 \\ &= 6\left(\frac{2}{3} - \frac{2}{3}w_2 - \frac{2}{3}w_1 - \frac{2}{3}x_3\right) + 3\left(\frac{1}{3} + \frac{2}{3}w_2 - \frac{1}{3}w_1 - \frac{1}{3}x_3\right) + 5x_3 \\ &= 5 - 2w_2 - 5w_1. \end{aligned}$$

This means that the same pivots as in **a)** will give an optimal dictionary for this objective function as well. The optimal value is 5, but the optimiser is not unique in this case, since the nonbasic variable x_3 is not present in the objective function. One sees from the optimal primal dictionary that x_3 can be increased to 1, so that the general optimiser is

$$(x_1, x_2, x_3) = \left(\frac{2}{3} - \frac{2}{3}x_3, \frac{1}{3} - \frac{1}{3}x_3, x_3\right) = \left(\frac{2}{3}, \frac{1}{3}, 0\right) + x_3\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$$

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for $0 \leq x_3 \leq 1$.

e) How is an extreme point of a convex set defined? Find at least three extreme points of the feasible region of the problem (1).
Solution: Denote the convex set by C. A point $x \in C$ is an extreme point if, whenever x can be written as a convex combination of two points $x^1, x^2 \in C$ (i.e., $x = (1-\lambda)x^1 + \lambda x^2$), we must have that $x = x^1 = x^2$.
Recall from "a mini-introduction to convexity" that the extreme points are the basic feasible solutions of (1). You can thus argue that

- $(0,0,0)$ is an extreme point since it is the basic solution we have at the start,
- $(0,0,1)$ is an extreme point since it is the basic solution you get by letting x_3 be the entering variable in the first iteration,
- $(2/3, 1/3, 0)$ is an extreme point since it is a basic solution for the optimal dictionary.

This gives three required extreme points.
One can also experiment by choosing any set of three nonbasic variables, and see if setting these to 0 gives a feasible point:
If $x_3 = w_1 = 0$ are nonbasic:

- If also $w_2 = 0$ we get that $x_1 + x_2 = 1$ and $\frac{1}{2}x_1 - x_2 = 0$, which gives the extreme point $(2/3, 1/3, 0)$, which we found in **a)**.
- If also $w_3 = 0$ we get that $x_1 + x_2 = 1$ and $-2x_1 + x_2 = 0$, which gives that $x_1 = 1/3$, $x_2 = 2/3$. $(1/3, 2/3, 0)$ is thus also an extreme point.

If $x_1 = x_2 = 0$ are nonbasic:

- If also $x_3 = 0$ we get the origin, which is feasible.
- If also $w_1 = 0$ we get that $x_3 = 1$, so we get the point $(0,0,1)$, which also is feasible.

These four are in fact all possible extreme points.
Geometric arguments from drawings of the feasible region are also allowed here.

Problem 2 (Convexity). Let $C \subseteq \mathbb{R}^{n+1}$ be a convex set, and let $D \subseteq \mathbb{R}^n$ be the set

$$D = \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x}, a) \in C \text{ for some } a \in \mathbb{R}\}.$$

D is also called the *projection* of C onto $\mathbb{R}^n$. Show that D is also convex.
Solution: Let $\mathbf{x}, \mathbf{y} \in D$. Then there exist $a, b \in \mathbb{R}$ so that $(\mathbf{x}, a) \in C$, and $(\mathbf{y}, b) \in C$. Since C is convex we have that

$$(1-\lambda)(\mathbf{x}, a) + \lambda(\mathbf{y}, b) = ((1-\lambda)\mathbf{x} + \lambda\mathbf{y}, (1-\lambda)a + \lambda b) \in C.$$

It follows that $(1-\lambda)\mathbf{x} + \lambda\mathbf{y} \in D$, so that D is also convex.

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Problem 3 (Game theory). Consider the matrix game with payoff matrix

$$A = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -1 & 0 \\ -1 & -1 & -2 \end{pmatrix}.$$

a) Find the pure minmax and maxmin strategies of the game. Does the matrix A have a saddle point? What does this say about the possibility of having pure strategies which are optimal?
Solution: The maxima in the rows are

$$P_R(1) = -1 \qquad P_R(2) = 0 \qquad P_R(3) = -1.$$

The minimum of these are $P_R(1) = -1$ and $P_R(3) = -1$, so that $i = 1$ and $i = 3$ are both pure minmax strategies, and $V^* = -1$.
The minima in the columns are

$$P_K(1) = -3 \qquad P_K(2) = -3 \qquad P_K(3) = -2.$$

The maximum of these are $P_K(3) = -2$, so that $j = 3$ is the pure maxmin strategy, and $V_* = -2$.
Since $V_* \neq V^*$ we know that A can't have a saddle point. This can also be checked directly by going through all entries in A, and verifying that it can not be both the maximum in its row and the minimum in its column. This also means that no pure strategies can be optimal, due to the minmax theorem.

b) Show that the randomised strategies $x^* = (1/2, 0, 1/2)$ and $y^* = (1/2, 0, 1/2)$ are mutually optimal for the column player and row player. What is the value of the game? Is the game fair?
Solution: We have that

$$Ax^* = \begin{pmatrix} -3/2 \\ -3/2 \\ -3/2 \end{pmatrix} \qquad (y^*)^T A = \begin{pmatrix} -3/2 & -2 & -3/2 \end{pmatrix}.$$

From this it follows that (since x and y are stochastic)

$$(y^*)^T Ax = -\frac{3}{2}x_1 - 2x_2 - \frac{3}{2}x_3$$
$$y^T Ax^* = -\frac{3}{2}y_1 - \frac{3}{2}y_2 - \frac{3}{2}y_3 = -\frac{3}{2}.$$

Since the maximum of $-\frac{3}{2}x_1 - 2x_2 - \frac{3}{2}x_3$ taken over all stochastic vectors is $-\frac{3}{2}$, it follows that the randomised strategies are mutually optimal, and that the value of the game is $-3/2$. The game is not fair since this is nonzero. This could also be inferred from the fact that the matrix have only nonpositive entries, so that all payments must go to the row player.

Problem 4 (Network flow). Consider the minimum cost network flow problem based on the directed graph shown in Figure 1. The number associated with each directed edge is the cost per unit flow, and the number associated with each node is the supply at that node.

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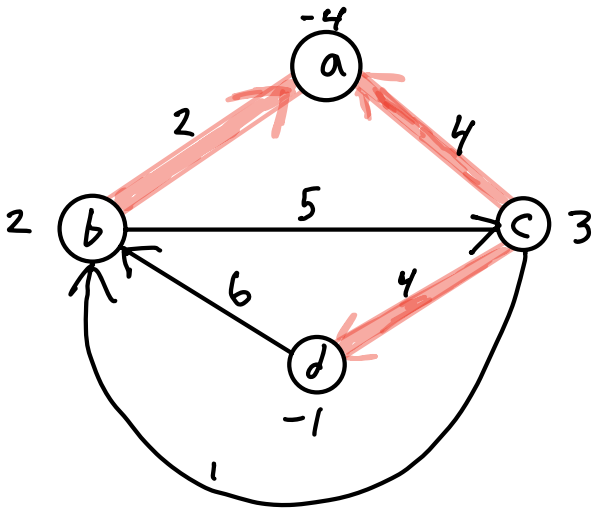


Figure 1: Flow problem for Problem 4.

a) Let T_1 be the spanning tree consisting of the edges (b, a), (c, a), and (c, d) (indicated in bold in the figure). Compute the tree solution corresponding to T_1.
Solution: We apply the flow balance equations at the nodes, starting with the two leaf nodes:

- Flow balance at b gives $x_{ba} = 2$.
- Flow balance at d gives $-x_{cd} = -1$, so that $x_{cd} = 1$.
- Flow balance at a gives $-x_{ba} - x_{ca} = -4$, so that $x_{ca} = 4 - x_{ba} = 2$.

We also have that $x_{bc} = x_{db} = x_{cb} = 0$, since these are outside T_1. The tree solution we found is thus (primal) feasible.

b) Use the network simplex method to find an optimal solution and optimal value for the flow problem.
Solution: Let b be the root node, so that $y_b = 0$. We first find the dual variables. We apply that $y_j - y_i = c_{ij}$ for each $(i, j) \in T_1$:

- $(i, j) = (b, a)$: $y_a - y_b = 2$, so that $y_a = 2$.
- $(i, j) = (c, a)$: $y_a - y_c = 4$, so that $y_c = -2$.
- $(i, j) = (c, d)$: $y_d - y_c = 4$, so that $y_d = 2$.

For the dual slack variables we apply that $y_j - y_i + z_{ij} = c_{ij}$ for each $(i, j) \notin T_1$:

- $(i, j) = (b, c)$: $y_c - y_b + z_{bc} = c_{bc}$, so that $-2 + z_{bc} = 5$, so that $z_{bc} = 7$.
- $(i, j) = (d, b)$: $y_b - y_d + z_{db} = c_{db}$, so that $-2 + z_{db} = 6$, so that $z_{db} = 8$.
- $(i, j) = (c, b)$: $y_b - y_c + z_{cb} = c_{cb}$, so that $2 + z_{cb} = 1$, so that $z_{cb} = -1$.

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Since z_{cb} is negative we do not have dual feasibility. We let x_{cb} enter, and increase it by ϵ. This gives a loop involving the three edges (c, b), (b, a), and (c, a). Due to the directions of the edges and flow balance, the flow is changed as follows within the loop:

$$\begin{aligned}\tilde{x}_{ba} &= x_{ba} + \epsilon = 2 + \epsilon \\ \tilde{x}_{ca} &= x_{ca} - \epsilon = 2 - \epsilon \\ \tilde{x}_{cb} &= \epsilon\end{aligned}$$

We see that $\tilde{x}_{ca}$ becomes zero first, and that this occurs for $\epsilon = 2$. x_{ca} thus leaves the basis. The new spanning tree is thus

$$T_2 = \{(b, a), (c, b), (c, d)\}.$$

The only changes for the new tree solution are $x_{ba} = 4$, $x_{cb} = 2$, $x_{ca} = 0$ (the flow can only change in the cycle introduced by the entering arc. In particular, $x_{cd} = 1$ still).
We compute the new dual variables:

- $(i, j) = (b, a)$: $y_a - y_b = 2$, so that $y_a = 2$.
- $(i, j) = (c, b)$: $y_b - y_c = 1$, so that $y_c = -1$.
- $(i, j) = (c, d)$: $y_d - y_c = 4$, so that $y_d = 3$.

The dual slacks for edges not in T_2 can be computed as follows:

- $(i, j) = (c, a)$: $y_a - y_c + z_{ca} = c_{ca}$, so that $2 + 1 + z_{ca} = 4$, so that $z_{ca} = 1$.
- $(i, j) = (b, c)$: $y_c - y_b + z_{bc} = c_{bc}$, so that $-1 + z_{bc} = 5$, so that $z_{bc} = 6$.
- $(i, j) = (d, b)$: $y_b - y_d + z_{db} = c_{db}$, so that $-3 + z_{db} = 6$, so that $z_{db} = 9$.

We thus have dual feasibility, and hence optimality. The optimal objective value is

$$\begin{aligned}&c_{ba}x_{ba} + c_{cb}x_{cb} + c_{cd}x_{cd} \\ &= 2 \cdot 4 + 1 \cdot 2 + 4 \cdot 1 \\ &= 8 + 2 + 4 = 14.\end{aligned}$$

Good luck!