LP. Lecture Game theory

Chapter 11: game theory

- matrix games
- optimal strategies
- von Neumann's minmax theorem
- connection to LP
- useful LP modeling of (certain) minmax and maxmin problems

Example: Paper-Scissors-Rock (= saks-papir-stein)

The game:

- Two persons independently choose one of the three options: Paper, Scissors or Rock
- Rules: Paper beats Rock, Rock beats Scissors, Scissors beats Paper.

Payoff matrix:

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

- Row player (R) chooses a row i, the Column player (K) chooses a column j, and the payoff is the entry a_{ij}: *the row player pays the column player a_{ij} kroner (NOK).*
- Similar for a general $m \times n$ matrix $A = [a_{ij}]$; this is called a Matrix game.

Pure strategies

- The choice above is called a pure strategy (or deterministic strategy): choose a row (or column). Note: in Paper-Scissors-Rock no pure strategy will be guaranteed to win (if the game is repeated), e.g., if R always chooses Paper K will soon choose Scissors.
- Goal: analyse Matrix games in general

Define

$P_R(i)$	$= \max_{j \leq n} a_{ij}$	: largest payoff for R using strategy i
$P_K(j)$	$= \min_{i \leq m} a_{ij}$	: smallest payoff for K using strategy j
V_*	$= \max_{j \leq n} P_K(j)$	: largest *guaranteed* payoff to K
V^*	$= \min_{i \leq m} P_R(i)$	: smallest *guaranteed* payoff from R

If $P_K(j) = V_*$ then j is called a pure maxmin strategy. If $P_R(i) = V^*$ then i is called a pure minmax strategy. If $V_* = V^*$, we say that the game has a value, namely $V := V_* = V^*$.

In Paper-Scissors-Rock: $V_* = -1 < 1 = V^*$.

Example: Consider the matrix game given by

$$A = \begin{bmatrix} 5 & 2 & 7 & 6 \\ 1 & 2 & 2 & 0 \\ 1 & 4 & 3 & 3 \end{bmatrix}$$

Then $P_K(1) = 1$, $P_K(2) = 2$, $P_K(3) = 2$, $P_K(4) = 0$, so $V_* = \max_j P_K(j) = 2$. Furthermore: $P_R(1) = 7$, $P_R(2) = 2$, $P_R(3) = 4$, so $V^* = \min_i P_R(i) = 2$.

Therefore $V_* = V^* = V = 2$. A pure maxmin strategy for K is $j = 3$ since $P_K(3) = 2 = V$, and a pure minmax strategy for R is $i = 2$ since $P_R(2) = 2 = V$.

Proposition

$$\text{(i)} \quad P_K(j) \leq a_{ij} \leq P_R(i) \qquad (i \leq m, j \leq n)$$

$$\text{(ii)} \quad P_K(j) \leq V_* \leq V^* \leq P_R(i) \quad (i \leq m, j \leq n)$$

Proof. $P_K(j) = \min_k a_{kj} \le a_{ij} \le \max_k a_{ik} = P_R(i)$. And (ii) follows from (i) by first taking max over j, which gives $P_K(j) \le V_* \le P_R(i)$ and then taking min over i; this gives (ii). ∎

A pair (r, s) of strategies (for R and K) is called a saddle point if

$$a_{rj} \le a_{rs} \le a_{is} \quad \text{for all } i, j$$

so r is the best choice for R when K chooses s, and s is the best choice for K when R chooses r. Note: a_{rs} is smallest in its column, and largest in its row.

Example: $(r, s) = (2, 1)$ is saddlepoint in

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}$$

In the example on the previous page both $(2, 2)$ and $(2, 3)$ are saddlepoints. Some matrices have a saddlepoint, others do not.

Theorem *The game has a value, player R has a pure minmax strategy r and player K has a pure maxmin strategy s if and only if (r, s) is a saddlepoint in A. In that case the value is $V = a_{rs}$.*

Proof. (i) Assume the game has a value V, player R has a pure minmax strategy r and player K has a pure maxmin strategy s. Then

$$a_{is} \geq P_K(s) = V_* = V = V^* = P_R(r) \geq a_{rj} \quad (i \leq m, j \leq n)$$

In particular, for $i = r$, $j = s$, we get $a_{rs} \geq V \geq a_{rs}$, so $V = a_{rs}$, and (again from the inequalities) $a_{is} \geq a_{rs} \geq a_{rj}$ for all i, j, which means that (r, s) is a saddlepoint.

(ii) Assume (r, s) is a saddlepoint, so $a_{rj} \leq a_{rs} \leq a_{is}$ for alle i, j Then

$$V_* = \max_j P_K(j) \geq P_K(s) = \min_i a_{is} = a_{rs}$$

and similarly $V^* = \min_i P_R(i) \leq P_R(r) = \max_j a_{rj} = a_{rs}$, so $V_* \geq V^*$. But, by the Proposition, $V_* = V^*$ and the equations imply that $V_* = V^* = a_{rs}$, r is a pure minmax strategy for R and s is a pure maxmin strategy for K. □

Randomized strategies

- The choice studied above is called a deterministic strategy: choose one row (or column).
- In Paper-Scissors-Rock no deterministic strategy can always win (if the game is played repeatedly), e.g., if R always chooses Paper, soon K will choose Scissors.
- May be better to use a randomized strategy: R chooses row i with probability y_i, and, *independently*, K chooses column j with probability x_j.
- So:

$$\sum_{i=1}^{m} y_i = 1, \quad y_i \geq 0 \quad (i \leq m)$$
$$\sum_{j=1}^{m} x_j = 1, \quad x_j \geq 0 \quad (j \leq n)$$

The Expected payoff from R to K is (recall probability theory!):

$$\sum_i \sum_j y_i a_{ij} x_j = y^T A x$$

Which strategy to use?

If player K chooses (randomized) strategy x, then the best choice for player R is to choose y so that $y^T A x$ is minimized (since R has to pay this amount). Therefore the best choice for K is to choose an x which is optimal in the problem

$$\max_x \min_y y^T A x$$

This is called a maxmin strategy.

Similarly analysis from player R's perspective: the best choice for R is a y which is optimal in the problem

$$\min_y \max_x y^T A x$$

This is called a minmax strategy.

In the (simple) Paper-Scissors-Rock game it follows from symmetry that $(1/3, 1/3, 1/3)$ is both a maxmin strategy (for K) and a minmax strategy (for R).

The maxmin problem: strategy for player K

Let e_i be the ith coordinate vector and e the all ones vector (of suitable size). Note that an LP with the feasible set being the standard simplex $S = \{y : \sum_i y_i = 1, y \geq O\}$ is easy, so we get:

$$v^* = \max_x \min_y y^T A x = \max_x \min_i e_i^T A x$$

Therefore player K's strategy problem may be written as the LP problem

$$\max\{v : v \leq e_i^T A x \;\; (i \leq m), \sum_j x_j = 1, x \geq O\}$$

with variables $v \in \mathbb{R},\ x \in \mathbb{R}^n$; or in matrix notation:

$$\begin{array}{lll} & \max & v \\ \text{(LP-K)} & \text{s.t.} & \\ & & ve - Ax \leq O \\ & & e^T x = 1 \\ & & x \geq O \end{array}$$

Thus: we can find an optimal strategy x for K efficiently by solving this LP.

The minmax problem: strategy for player R

Similar analysis for player R:

$$u^* = \min_y \max_x y^T A x = \min_y \max_j y^T A e_j$$

So, player R's strategy problem becomes the LP problem

$$\min\{u : u \geq y^T A e_j \ (j \leq n), \sum_i y_i = 1, y \geq O\}$$

with variables $u \in \mathbb{R},\ y \in \mathbb{R}^m$; which is

$$\text{(LP-R)} \quad \begin{array}{ll} \min & u \\ \text{s.t.} & \\ & ue - A^T y \geq O \\ & e^T y = 1 \\ & y \geq O \end{array}$$

The minmax theorem

Theorem [John von Neumann(1928)] *Let x^* be an optimal strategy for player K and y^* an optimal strategy for player R. Then*

$$v^* = \max_x (y^*)^T A x = \min_y y^T A x^* = u^*$$

i.e., $\min_y \max_x y^T A x = \max_x \min_y y^T A x$.

Proof. One can check that problem LP-R is the dual LP of problem LP-K. (Exercise!) So, by the duality theorem of LP the optimal value v^* of LP-K equals the optimal value u^* of LP-R, and this proves the theorem. □

- The common value $v^* = u^*$ is called the value of the game: this is the expected payoff when both players play optimally
- It is also possible to prove the LP duality theorem from von Neumann's theorem
- Solve the LP's above, for some selected A's, using OPL-CPLEX.