Answers to Exercises, Week 10, MAT3100, V20

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Exercises in Week 10 are: 1-9 from 'A mini-introduction to convexity' by Geir Dahl.

Exercise 1

To show $B(0,1)$ is convex. Let $x, y \in B(0,1)$. Then $\|x\|, \|y\| \leq 1$. Let $\lambda \in [0,1]$ and $z = (1-\lambda)x + \lambda y$. Then

$$\|z\| \leq (1-\lambda)\|x\| + \lambda\|x\|y$$

by the triangle inequalty, and so

$$\|z\| \leq (1-\lambda) + \lambda = 1.$$

Therefore $z \in B(0,1)$.

The argument for $B(a,r)$ is similar. If $\|x-a\| \leq r$ and $\|y-a\| \leq r$ then

$$\|((1-\lambda)x + \lambda y) - a\| = \|(1-\lambda)(x-a) + \lambda(y-a)\| \leq r$$

for $\lambda \in [0,1]$.

Exercise 2

Let L be a linear subspace of $\mathbb{R}^n$. Let $x, y \in L$, $\lambda \in [0,1]$. Since L is a linear space, $(1-\lambda)x \in L$ and $\lambda y \in L$ and then further, $(1-\lambda)x + \lambda y \in L$.

Exercise 3

The union of convex sets might not be convex. For example, the union of two distinct points is not.

Exercise 4

Suppose $C_1, \ldots, C_t \in \mathbb{R}^n$ are convex and let $I = C_1 \cap C_2 \cap \cdots \cap C_t$. Let $x, y \in I$, $\lambda \in [0,1]$. Then since $x, y \in C_i$ all i, and C_i is convex, $(1-\lambda)x + \lambda y \in C_i$. Therefore $(1-\lambda)x + \lambda y \in I$.

Exercise 5

Suppose $B = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ is a polyhedron. Since it's bounded it must be a polytope, by Corollary 6. Thus it must have a finite number of extreme points. Let $x \in B$ with $\|x\| = 1$. Let us show that x is an extreme point. We will show that we cannot express x as $x = (1/2)x^1 + (1/2)x^2$ for distinct points x^1 and x^2 in B. To get a contradiction, suppose that $x = (1/2)x^1 + (1/2)x^2$ for distinct points x^1 and x^2 in B. Then

$$1 = \|x\| \leq (1/2)\|x^1\| + (1/2)\|x^2\| \leq (1/2) + (1/2) = 1.$$

Therefore, $\|x^1\| = \|x^2\| = 1$ and the first inequality is an equality which implies that $x^2 = \mu x^1$ for some $\mu \in \mathbb{R}$. Then $1 = \|x^2\| = |\mu|\|x^1\| = |\mu|$ and so $x^2 = \pm x^1$. If $x^2 = -x^1$ then $x = 0$ which is a contradiction. If $x^2 = x^1$ then x^1 and x^2 are not distinct, again a contradiction. Thus B has infinitely many extreme points and connot be a polyhedron.

Exercise 6

Suppose $B_\infty = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. Then

$$\begin{aligned} B_\infty &= \{x \in \mathbb{R}^n : \max_j |x_j| \leq 1\} \\ &= \{x \in \mathbb{R}^n : |x_j| \leq 1, j = 1, \ldots, n\} \\ &= \{x \in \mathbb{R}^n : -1 \leq x_j \leq 1, j = 1, \ldots, n\} \end{aligned}$$

which is a polyhedron because it's the set of solutions to a finite set of linear inequalities. In fact it's the unit cube $[-1, 1]^n$.

Exercise 7

Suppose $B_1 = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$. Let

$$E_n = \{(\epsilon_1, \ldots, \epsilon_n) : \epsilon_i \in \{-1, 0, 1\}, i = 1, \ldots, n\}.$$

Then the inequality

$$\sum_{j=1}^{n} |x_j| \leq 1 \tag{1}$$

is equivalent to the system of inequalities

$$\sum_{j=1}^{n} \epsilon_j x_j \leq 1 \quad \text{for all} \quad \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in E_n. \tag{2}$$

Proof: Let x be a solution to (1). Then for $\epsilon \in E_n$,

$$\sum_{j=1}^{n} \epsilon_j x_j \le \left| \sum_{j=1}^{n} \epsilon_j x_j \right| \le \sum_{j=1}^{n} |\epsilon_j||x_j| \le \sum_{j=1}^{n} |x_j| \le 1.$$

For the converse, let x be a solution to (2). Let

$$\tilde{\epsilon} = (\mathrm{sgn}(x_1), \mathrm{sgn}(x_2), \ldots, \mathrm{sgn}(x_n)).$$

Then

$$\sum_{j=1}^{n} |x_j| = \sum_{j=1}^{n} \tilde{\epsilon}_j x_j \le 1.$$

Now

$$B_1 = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} \epsilon_j x_j \le 1, \, \epsilon \in E_n\},$$

which is a polyhedron because it's the set of solutions to a finite set of linear inequalities.

Exercise 8

We are given the LP problem

$$\begin{aligned} \max \quad & c^T x, \\ \text{subj.} \quad & Ax \le b. \end{aligned}$$

First we put it in standard form. Set $x_i = x_i^+ - x_i^-$, $x_i^-, x_i^+ \ge 0$. Then $x = x^+ - x^-$, $x^-, x^+ \ge 0$. Let

$$\tilde{c} = \begin{bmatrix} c \\ -c \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & -A \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}.$$

Then the LP problem becomes

$$\begin{aligned} \max \quad & \tilde{c}^T \tilde{x}, \\ \text{subj.} \quad & \tilde{A}\tilde{x} \le b, \\ & \tilde{x} \ge 0. \end{aligned}$$

Now introduce the slack vector $w = b - \tilde{A}\tilde{x}$. Let

$$c_1 = \begin{bmatrix} \tilde{c} \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \tilde{A} & I \end{bmatrix}, \quad x_1 = \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}.$$

Then the LP problem becomes

$$\begin{aligned} \max \quad & c_1^T x_1, \\ \text{subj.} \quad & A_1 x_1 = b, \\ & x_1 \geq 0. \end{aligned}$$

Exercise 9

Let

$$P = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1,\ i = 1, \ldots, n\}.$$

We are asked to solve the LP problem

$$\max\{c^T x : x \in P\}.$$

It is sufficient to maximize the objective function over the extreme points of P. These are the vertices

$$V = \{v = (v_1, \ldots, v_n) : v_i \in \{0, 1\}, i = 1, \ldots, n\}.$$

So the problem reduces to

$$\max\{c^T v = \sum_{i=1}^{n} c_i v_i : v \in V\}.$$

We can maximize each term $c_i v_i$ independently. If $c_i > 0$ we must set $v_i = 1$ and if $c_i < 0$ we must set $v_i = 0$. If $c_i = 0$ we can take either $v_i = 0$ or $v_i = 1$. Let $I^+ = \{i : c_i > 0\}$, $I^0 = \{i : c_i = 0\}$, and $I^- = \{i : c_i < 0\}$. Then an optimal solution is

$$x^* = (v_1, \ldots, v_n)$$

where $v_i = 1$ if $i \in I^+$, $v_i = 0$ if $i \in I^-$, and $v_i \in \{0, 1\}$ if $i \in I^0$. The optimal value of the objective function is

$$\sum_{i \in I^+} c_i.$$

If $I^0 \neq \emptyset$ the solution x^* is not unique. We can take convex combinations and obtain the solutions

$$x^* = (x_1, \ldots, x_n)$$

where $x_i = 1$ if $i \in I^+$, $x_i = 0$ if $i \in I^-$, and $0 \leq x_i \leq 1$ if $i \in I^0$. If $\#(I^0) = 1$, the solution set is an edge (a 1-face), if $\#(I^0) = 2$, the solution set is a 2-face, etc. and if $\#(I^0) = n$, the solution set is the whole of P.

Next we generalize to

$$P = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i,\, i = 1, \ldots, n\}.$$

Then we can maximize over

$$V = \{v = (v_1, \ldots, v_n) : v_i \in \{a_i, b_i\}, i = 1, \ldots, n\}.$$

Again we can maximize each term $c_i v_i$ independently. We must take $v_i = b_i$ if $c_i > 0$, and $v_i = a_i$ if $c_i < 0$. If $c_i = 0$ we can take either. We now obtain an optimal solution

$$x^* = (v_1, \ldots, v_n)$$

where $v_i = b_i$ if $i \in I^+$, $v_i = a_i$ if $i \in I^-$, and $v_i \in \{a_i, b_i\}$ if $i \in I^0$. The optimal value of the objective function is

$$\sum_{i \in I^+} c_i b_i + \sum_{i \in I^-} c_i a_i.$$

As before, we can take convex combinations of solution points if $I^0 \neq \emptyset$.