Answers to Exercises, Week 11, MAT3100, V20

Michael Floater

Answers to Exercises 10-18 from 'A mini-introduction to convexity' by Geir Dahl.

Exercise 10

Show that conv(S) is convex for any $S \subset \mathbb{R}^n$. Answer: conv(S) consists of all points of the form

$$x = \sum_{j=1}^{m} \lambda_j x_j$$

where $m \geq 1$, $x_1, x_2, \ldots, x_m \in S$, $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ and $\sum_{j=1}^{m} \lambda_j = 1$.

Let $x, y \in \text{conv}(S)$. Then we can express x as above, and y as

$$y = \sum_{j=m+1}^{n} \mu_j x_j$$

where $n \geq m+1$, $x_{m+1}, x_{m+2}, \ldots, x_n \in S$, $\mu_{m+1}, \mu_{m+2}, \ldots, \mu_n \geq 0$ and $\sum_{j=m+1}^{n} \mu_j = 1$. Then, for any $\alpha \in \mathbb{R}$,

$$(1-\alpha)x + \alpha y = \sum_{j=1}^{m} (1-\alpha)\lambda_j x_j + \sum_{j=m+1}^{n} \alpha \mu_j x_j = \sum_{j=1}^{n} \nu_j x_j,$$

where

$$\nu_j = \begin{cases} (1-\alpha)\lambda_j, & 1 \leq j \leq m; \\ \alpha \mu_j & m+1 \leq j \leq n, \end{cases}$$

and

$$\sum_{j=1}^{n} \nu_j = (1-\alpha) \sum_{j=1}^{m} \lambda_j + \sum_{j=m+1}^{n} \alpha \mu_j = (1-\alpha) + \alpha = 1.$$

If also $\alpha \in [0,1]$, then $\nu_{m+1}, \nu_{m+2}, \ldots, \nu_n \geq 0$, and so $(1-\alpha)x + \alpha y \in \text{conv}(S)$.

Exercise 11

Two distinct sets in $\mathbb{R}^1$ with the same convex hull are $S = \{0, 1, 2\}$ and $T = \{0, 2\}$. Their common convex hull is $\text{conv}(S) = \text{conv}(T) = [0, 2]$.

Exercise 12

Show that if $S \subseteq T$ then $\text{conv}(S) \subseteq \text{conv}(T)$. Answer: suppose $S \subseteq T$ and let $x \in \text{conv}(S)$. Then

$$x = \sum_{j=1}^{m} \lambda_j x_j$$

where $m \geq 1$, $x_1, x_2, \ldots, x_m \in S$, $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ and $\sum_{j=1}^{m} \lambda_j = 1$. Then $x_1, x_2, \ldots, x_m \in T$, and so $x \in \text{conv}(T)$.

Exercise 13

Show that if S is convex then $\text{conv}(S) = S$. Answer: it is trivial that $S \subseteq \text{conv}(S)$ (because every point in S is a convex combination of itself). Conversely, to show that $\text{conv}(S) \subseteq S$, let $x \in \text{conv}(S)$. Then

$$x = \sum_{j=1}^{m} \lambda_j x_j$$

where $m \geq 1$, $x_1, x_2, \ldots, x_m \in S$, $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ and $\sum_{j=1}^{m} \lambda_j = 1$. Then, because S is convex, $x \in S$.

Exercise 14

Let $S = \{x \in \mathbb{R}^2 : \||x\|_2 = 1\}$, the unit circle. Determine $\text{conv}(S)$. Answer: let $B = \{x \in \mathbb{R}^2 : \||x\|_2 \leq 1\}$, the unit ball. We claim that $\text{conv}(S) = B$.

To show this, using the results of the previous exercises, since $S \subseteq B$ and B is convex,

$$\text{conv}(S) \subseteq \text{conv}(B) = B.$$

Conversely, if $x \in B$ then we can show that $x \in \text{conv}(S)$. If $\|x\| = 1$ then $x \in \text{conv}(S)$ trivially. Otherwise, $\|x\| < 1$. Then we can pass any straight line through x and it will intersect S at two points $y, z \in S$. Then there is some $\lambda \in (0, 1)$ such that $x = (1 - \lambda)y + \lambda z$, and so $x \in \text{conv}(S)$. Therefore $B \subseteq \text{conv}(S)$.

Exercise 15

Let $S = \{(0,0),(1,0),(0,1)\}$. Then $\text{conv}(S)$ consists of all points

$$x = x_0(0,0) + x_1(1,0) + x_2(1,1),$$

where $x_0, x_1, x_2 \geq 0$ and $x_0 + x_1 + x_2 = 1$. Then $x = (x_1, x_2)$ and $x_1, x_2 \geq 0$ and $x_1 + x_2 \leq 1$. Therefore,

$$\text{conv}(S) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0,\ x_1 + x_2 \leq 1\}.$$

Exercise 16

Let

$$S = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}.$$

Show that $\text{conv}(S) = B$, where

$$B = \{(x_1,x_2,x_3) : 0 \leq x_i \leq 1, i = 1,2,3\}.$$

Answer. First let $x \in \text{conv}(S)$. Then

$$\begin{aligned} x = &\ \lambda_1(0,0,0) + \lambda_2(1,0,0) + \lambda_3(0,1,0) + \lambda_4(1,1,0) \\ &+ \lambda_5(0,0,1) + \lambda_6(1,0,1) + \lambda_7(0,1,1) + \lambda_8(1,1,1), \end{aligned}$$

where $\sum_{j=1}^{8} \lambda_j = 1$ and $\lambda_j \geq 0$, $j = 1,2,\ldots,8$. Then

$$x = (\lambda_2 + \lambda_4 + \lambda_6 + \lambda_8, \lambda_3 + \lambda_4 + \lambda_7 + \lambda_8, \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8),$$

and so $x \in B$. Therefore, $\text{conv}(S) \subseteq B$.

Conversely, suppose $x = (x_1,x_2,x_3) \in B$. Define

$$\lambda_i(u) = u^i(1-u)^{1-i}, \quad u \in \mathbb{R}, \quad i = 0,1,$$

and

$$B_{i,j,k}(x) = \lambda_i(x_1)\lambda_j(x_2)\lambda_k(x_3), \quad i,j,k \in \{0,1\},$$

and

$$v_{i,j,k} = (i,j,k), \quad i,j,k \in \{0,1\}.$$

For $u \in \mathbb{R}$,

$$\sum_{i=0}^{1} \lambda_i(u) i = u,$$

and

$$\sum_{i=0}^{1} \lambda_i(u) = (1-u) + u = 1,$$

and therefore

$$x = \sum_{i,j,k=0}^{1} B_{i,j,k}(x) v_{i,j,k},$$

and

$$1 = \sum_{i,j,k=0}^{1} B_{i,j,k}(x).$$

Since also $B_{i,j,k}(x) \geq 0$ for $i, j, k \in \{0, 1\}$, it follows that $x \in \text{conv}(S)$.

Exercise 17

Show a different version of Farkas's lemma: show that $Ax = b$ has a non-negative solution if and only if $y^T b \geq 0$ for all y s.t. $y^T A \geq 0$. Here, $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$.

To answer this we can use a similar proof to the one in the notes. We consider the pair of dual LP problems:

$$\text{(P)} \max c^T x \text{ subj. to } Ax = b \text{ and } x \geq 0,$$

and

$$\text{(D)} \min y^T b \text{ subj. to } A^T y \leq c.$$

(Here y is free). Now let $c = 0$. Then (D) is feasible since $y = 0$ is feasible. So by LP theory (D) either has an optimal solution or is unbounded. If it is unbounded, the minimum of its objective value must be $-\infty$ and by weak duality, (P) cannot have a feasible solution. In this case there is no $x \geq 0$ with $Ax = b$. Otherwise (D) has an optimal solution. Then we must have $y^T b \geq 0$ for all y s.t. $y^T A \leq 0$. This is because if there were a y s.t. $y^T b < 0$ and $y^T A \leq 0$, then by considering points $y' = \lambda y$, $\lambda > 0$, we see that (D) would be unbounded.

Exercise 18

Part (i) was x solved in the text. In Part (ii),

$$P = \{x \in \mathbb{R}^4 : 2x_1 + 3x_2 = 12, x_1 + x_2 + x_3 + x_4 = 1, x \geq 0\}.$$

Then

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 12 \\ 1 \end{bmatrix}.$$

Extrema of A correspond to invertible submatrices of A that give a feasible solution.

$$\begin{aligned}
&\{1,2\}, \quad A_B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, \quad x_B = A_B^{-1} b = \begin{bmatrix} -9 \\ 10 \end{bmatrix}, \quad \text{not feasible,} \\
&\{1,3\}, \quad A_B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad x_B = A_B^{-1} b = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad \text{not feasible,} \\
&\{1,4\}, \quad A_B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad x_B = A_B^{-1} b = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad \text{not feasible,} \\
&\{2,3\}, \quad A_B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \quad x_B = A_B^{-1} b = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \quad \text{not feasible,} \\
&\{2,4\}, \quad A_B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \quad x_B = A_B^{-1} b = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \quad \text{not feasible,} \\
&\{3,4\}, \quad A_B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{not invertible.}
\end{aligned}$$

There are no extrema. In fact the polyedron is empty.