Answers to Exercises, Week 1, MAT3100, V20

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Exercises in Week 1 are: 1.1, 2.1, 2.2 of Vanderbei.

Exercise 1.1

Let x_b be the number of tons of bands to make, $x_b \geq 0$, and x_c be the number of tons of coils to make, $x_c \geq 0$. The profit will then be $25x_b + 30x_c$. The limit on the numbers of hours of production time is $40h$. The number of hours required to make one ton of bands is $(1/200)h$, and the number of hours required to make one ton of coils is $(1/140)h$. So the time constraint is

$$\frac{1}{200}x_b + \frac{1}{140}x_c \leq 40.$$

The upper bounds set by the company are $x_b \leq 6000$ and $x_c \leq 4000$.

Thus the linear programming problem is:

$$\begin{array}{lrl} \text{maximize} & 25x_b + 30x_c & \\ \text{subject to} & (1/200)x_b + (1/140)x_c & \leq 40, \\ & x_b & \leq 6000, \\ & x_c & \leq 4000, \\ & x_b, x_c & \geq 0. \end{array}$$

It's possible to solve this problem by inspection. Since we have only two variables we can determine the feasible region, the set of feasible solutions (x_b, x_c), 'by hand'. The feasible region in this case is the intersection of the rectangle $[0, 6000] \times [0, 4000]$ and the half plane below the line

$$\frac{1}{200}x_b + \frac{1}{140}x_c = 40.$$

This is a five-sided polygon. After some calculations we find that the five vertices of this polygon are, in anticlockwise order,

$$(0, 0), \quad (6000, 0), \quad (6000, 1400), \quad (16000/7, 4000) \quad (0, 4000).$$

Since the objective function $\eta(x_b, x_c)$ is linear in x_b and x_c, its maximal value in the feasible region must be attained at one of these five vertices. We find

$$\begin{aligned}
\eta(0,0) &= 0, \\
\eta(6000,0) &= 150000, \\
\eta(6000,1400) &= 192000, \\
\eta(16000/7,4000) &= 1240000/7 \approx 177142, \\
\eta(0,4000) &= 120000.
\end{aligned}$$

The largest of these five values is 192000 and so the optimal solution is $(x_b, x_c) = (6000, 1400)$ and the optimal value (profit in this case) is $\eta = 192000$.

Exercise 2.1

The LP problem is

$$\begin{array}{rrl}
\text{maximize} & 6x_1 + 8x_2 + 5x_3 + 9x_4 & \\
\text{subject to} & 2x_1 + x_2 + x_3 + 3x_4 & \leq 5, \\
& x_1 + 3x_2 + x_3 + 2x_4 & \leq 3, \\
& x_1, x_2, x_3, x_4 & \geq 0.
\end{array}$$

We introduce the slack variables

$$\begin{aligned}
w_1 &= 5 - 2x_1 - x_2 - x_3 - 3x_4, \\
w_2 &= 3 - x_1 - 3x_2 - x_3 - 2x_4,
\end{aligned}$$

and then the initial dictionary is

$$\begin{array}{ccccccccccc}
\eta & = & & & 6x_1 & + & 8x_2 & + & 5x_3 & + & 9x_4 \\
\hline
w_1 & = & 5 & - & 2x_1 & - & x_2 & - & x_3 & - & 3x_4 \\
w_2 & = & 3 & - & x_1 & - & 3x_2 & - & x_3 & - & 2x_4
\end{array}$$

The initial solution is $x_1 = x_2 = x_3 = x_4 = 0$ and $w_1 = 5$ and $w_2 = 3$. The initial basic variables are w_1, w_2. The initial basis is $\{w_1, w_2\}$. The remaining variables are non-basic, initially x_1, x_2, x_3, x_4. The non-basic variables always have value 0.

We can increase any variable in the objective function if it has a positive coefficient. So we could here increase any one of x_1, x_2, x_3, x_4. Since x_4

has the largest positive coefficient, 9, it is a reasonable idea to choose to increase x_4. How much can we increase x_4 while keeping $x_1 = x_2 = x_3 = 0$? We require

$$\begin{aligned} w_1 &= 5 - 3x_4 \geq 0, \\ w_2 &= 3 - 2x_4 \geq 0, \end{aligned}$$

and so the maximal new value of x_4 is

$$\min\left\{\frac{5}{3}, \frac{3}{2}\right\} = \frac{3}{2},$$

and w_2 is the variable that becomes zero when we set $x_4 = 3/2$. Thus we perform a pivot: we take x_4 into the basis and take w_2 out of the basis.

How do we carry out this pivot? First we use the equation for w_2 to express x_4 in terms of w_2 and the other non-basic variables, i.e.,

$$x_4 = 3/2 - (1/2)x_1 - (3/2)x_2 - (1/2)x_3 - (1/2)w_2.$$

Then we substitute this expression for x_4 into the equation for w_1:

$$\begin{aligned} w_1 &= 5 - 2x_1 - x_2 - x_3 - 3x_4 \\ &= 5 - 2x_1 - x_2 - x_3 - 3(3/2 - (1/2)x_1 - (3/2)x_2 - (1/2)x_3 - (1/2)w_2) \\ &= (1/2) - (1/2)x_1 + (7/2)x_2 + (1/2)x_3 + (3/2)w_2. \end{aligned}$$

Alternatively, we can use an elementary row operation: we can get this new equation for w_1 by substracting 3/2 times the equation for w_2 in the initial dictionary from the equation for w_1 in the initial dictionary.

Using either approach we can similarly get a new expression η. Using substitutuion we get

$$\begin{aligned} \eta &= 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ &= 6x_1 + 8x_2 + 5x_3 + 9(3/2 - (1/2)x_1 - (3/2)x_2 - (1/2)x_3 - (1/2)w_2) \\ &= (27/2) + (3/2)x_1 - (11/2)x_2 + (1/2)x_3 - (9/2)w_2. \end{aligned}$$

Alternatively, we can use elementary row operations: for example, we could obtain the new equation for w_1 by substracting 3/2 times the equation for w_2 in the initial dictionary from the equation for w_1 in the initial dictionary.

The new dictionary is

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & (27/2) & + & (3/2)x_1 & - & (11/2)x_2 & + & (1/2)x_3 & - & (9/2)w_2 \\
\hline
w_1 & = & (1/2) & - & (1/2)x_1 & + & (7/2)x_2 & + & (1/2)x_3 & + & (3/2)w_2 \\
x_4 & = & (3/2) & - & (1/2)x_1 & - & (3/2)x_2 & - & (1/2)x_3 & - & (1/2)w_2
\end{array}
$$

Note that the current value of η is now 27/2. The basic variables have value $w_1 = 1/2$, $x_4 = 3/2$. The non-basic variables are x_1, x_2, x_3, w_2.

We now repeat the process. We look for positive coefficients in η in the current dictionary. We can increase any variable with a positive coefficient. Here, x_1 and x_3 have positive coefficients. We may as well increase x_1 since it has the largest coefficient. How much can we increase x_1 while keeping the other non-basic variables equal to 0 ($x_2 = x_3 = w_2 = 0$)? We require

$$
\begin{aligned}
w_1 &= (1/2) - (1/2)x_1 \geq 0, \\
x_4 &= (3/2) - (1/2)x_1 \geq 0,
\end{aligned}
$$

and so the maximal new value of x_1 is

$$\min\{1, 3\} = 1,$$

and w_1 is the variable that becomes zero when we set $x_1 = 1$. Thus we perform a pivot: we take x_1 into the basis and take w_1 out of the basis. Proceeding as before we get the new dictionary

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 15 & - & 3w_1 & + & 5x_2 & + & 2x_3 & & \\
\hline
x_1 & = & 1 & - & 2w_1 & + & 7x_2 & + & x_3 & + & 3w_2 \\
x_4 & = & 1 & + & w_1 & - & 5x_2 & - & x_3 & - & 2w_2
\end{array}
$$

The value of η has now increased to 15.

Now we could increase either x_2 or x_3. If we choose to increase x_2 we require

$$
\begin{aligned}
x_1 &= 1 + 7x_2 \geq 0, \\
x_4 &= 1 - 5x_2 \geq 0.
\end{aligned}
$$

The first inequality holds for any $x_2 \geq 0$. Just the second inequality puts a constraint on x_2: we require that $x_2 \leq 1/5$. Thus we put x_2 into the basis and take x_4 out and we get the new dictionary:

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 16 & - & 2w_1 & - & x_4 & + & x_3 & - & 2w_2 \\
\hline
x_1 & = & (12/5) & - & (3/5)w_1 & - & (7/5)x_4 & - & (2/5)x_3 & + & (1/5)w_2 \\
x_2 & = & (1/5) & + & (1/5)w_1 & - & (1/5)x_4 & - & (1/5)x_3 & - & (2/5)w_2
\end{array}
$$

The value of η has now increased to 16. The only variable we can increase now is x_3. We can increase it to 1 and x_2 becomes zero. So we take x_3 into the basis and remove x_2 and we get

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 17 & - & w_1 & - & 2x_4 & - & 5x_2 & - & 4w_2 \\
\hline
x_1 & = & 2 & - & w_1 & - & x_4 & + & 2x_2 & + & w_2 \\
x_3 & = & 1 & + & w_1 & - & x_4 & - & 5x_2 & - & 2w_2
\end{array}
$$

The value of η has now increased to 17. We can no longer increase η because all its coefficients are non-positive. So we have reached the optimal solution and

$$x_1 = 2, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 0.$$

The values of the slack variables are in this case zero, $w_1 = w_2 = 0$. So in this LP problem the two constraints are fulfilled exactly, but this will not always be the case.

We used four iterations in the simplex algorithm. In hindsight, now that we know the optimal solution, it might have better to have increased either x_1 or x_3 at the first step. We might have reached the optimal solution in two steps instead of four.

Exercise 2.2

The LP problem is

$$
\begin{array}{rrl}
\text{maximize} & 2x_1 + x_2 & \\
\text{subject to} & 2x_1 + x_2 & \leq 4, \\
 & 2x_1 + 3x_2 & \leq 3, \\
 & 4x_1 + x_2 & \leq 5, \\
 & x_1 + 5x_2 & \leq 1, \\
 & x_1, x_2 & \geq 0.
\end{array}
$$

We introduce slack variables, one for each constraint:

$$\begin{aligned}
w_1 &= 4 - 2x_1 - x_2, \\
w_2 &= 3 - 2x_1 - 3x_2, \\
w_3 &= 5 - 4x_1 - x_2, \\
w_4 &= 1 - x_1 - 5x_2,
\end{aligned}$$

and then the initial dictionary is

$$\begin{array}{ccccccc}
\eta & = & & & 2x_1 & + & x_2 \\
\hline
w_1 & = & 4 & - & 2x_1 & - & x_2 \\
w_2 & = & 3 & - & 2x_1 & - & 3x_2 \\
w_3 & = & 5 & - & 4x_1 & - & x_2 \\
w_4 & = & 1 & - & x_1 & - & 5x_2
\end{array}$$

The initial solution is $x_1 = x_2 = 0$ and $(w_1, w_2, w_3, w_4) = (4, 3, 5, 1)$.

Both x_1 and x_2 have positive coefficients and so we can increase either of them. x_1 has the largest positive coefficient, 2, so let's try that. How much can we increase x_1 while keeping $x_2 = 0$? We require

$$\begin{aligned}
w_1 &= 4 - 2x_1 \geq 0, \\
w_2 &= 3 - 2x_1 \geq 0, \\
w_2 &= 5 - 4x_1 \geq 0, \\
w_2 &= 1 - x_1 \geq 0,
\end{aligned}$$

and so the maximal new value of x_1 is

$$\min\left\{\frac{4}{2}, \frac{3}{2}, \frac{5}{4}, \frac{1}{1}\right\} = 1,$$

and w_4 is the variable that becomes zero when we set $x_1 = 1$. Thus we perform a pivot, putting x_1 into the basis and taking w_4 out. The new dictionary is

$$
\begin{array}{rcrcrcr}
\eta & = & 2 & - & 2w_4 & - & 9x_2 \\
\hline
w_1 & = & 2 & + & 2w_4 & + & 9x_2 \\
w_2 & = & 1 & + & 2w_4 & + & 7x_2 \\
w_3 & = & 1 & + & 4w_4 & + & 19x_2 \\
x_1 & = & 1 & - & w_4 & - & 5x_2
\end{array}
$$

Since both coefficients in η are now non-positive, we have reached the optimal solution, which is $x_1 = 1$ and $x_2 = 0$ with the optimal value of the objective function $\eta = 2$. The values of the slack variables are $(w_1, w_2, w_3) = (2, 1, 1)$ and $w_4 = 0$. So in the optimal solution, the fourth constraint is fulfilled exactly, but not the others.