Answers to Exercises, Week 4, MAT3100, V20

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Exercises in Week 4 are: Ex. 2.16, 2.18, 3.1, 3.2, 3.4 of Vanderbei. Optional extra: 2.19.

Exercise 2.16

The feasible region is shown in Figure 1.

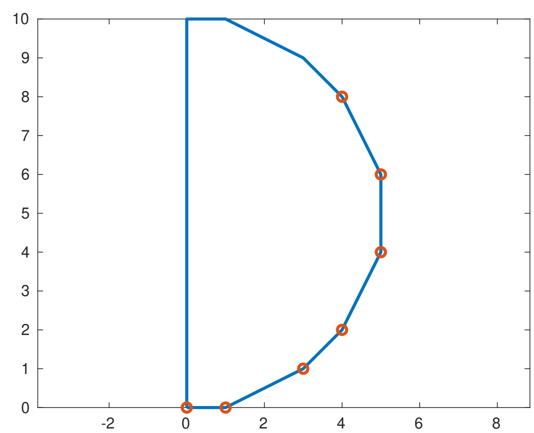

Figure 1: Feasible region.

The optimal solution is the vertex $(x_1, x_2) = (4, 8)$. The initial feasible solution is $(x_1, x_2) = (0, 0)$. If we use the largest coefficient rule to decide on the first pivot, we will increase x_1, and the simplex method will make the sequence of pivots (iterations) shown in the figure, going from vertex to vertex. If instead we increase x_2 in the first iteration the method will go around the polygon in the other direction.

Exercise 2.18

The current dictionary can be expressed as

$$\begin{aligned}
\eta &= d + \sum_{j \in N} c_j x_j, \\
x_i &= b_i - \sum_{j \in N} a_{ij} x_j, \qquad i \in B,
\end{aligned}$$

where $b_i \geq 0$, $i \in B$. Note here that in general the a's, b's, and c's will be different to those in the LP problem we started with.

We can let x_l, $l \in N$, enter the basis if $c_l > 0$. We can then let x_k, $k \in B$, leave the basis if $a_{kl} > 0$ and

$$\frac{b_k}{a_{kl}} = \min \left\{ \frac{b_i}{a_{il}} : i \in B, a_{il} > 0 \right\}.$$

Let us then compute the next dictionary. We start with the equation with $i = k$ and write it as

$$x_k = b_k - \sum_{j \in N \setminus l} a_{kj} x_j - a_{kl} x_l,$$

and then invert the roles of x_k and x_l:

$$x_l = \frac{1}{a_{kl}} \left(b_k - \sum_{j \in N \setminus l} a_{kj} x_j - x_k \right).$$

We then substitute this expression into the other rows of the dictionary. For $i \in B$, $i \neq k$,

$$\begin{aligned}
x_i &= b_i - \sum_{j \in N \setminus l} a_{ij} x_j - a_{il} x_l \\
&= b_i - \sum_{j \in N \setminus l} a_{ij} x_j - \frac{a_{il}}{a_{kl}} \left(b_k - \sum_{j \in N \setminus l} a_{kj} x_j - x_k \right) \\
&= b_i^* - \sum_{j \in N^*} a_{ij}^* x_j,
\end{aligned}$$

where $N^* = (N \setminus l) \cup k$, and

$$\begin{aligned} b_i^* &= b_i - \frac{a_{il}}{a_{kl}} b_k, \\ a_{ij}^* &= a_{ij} - \frac{a_{il}}{a_{kl}} a_{kj}, \quad j \neq k, \\ a_{ik}^* &= -\frac{a_{il}}{a_{kl}}. \end{aligned}$$

For η we get

$$\eta = d^* + \sum_{j \in N^*} c_j^* x_j,$$

where

$$\begin{aligned} d^* &= d + \frac{c_l}{a_{kl}} b_k, \\ c_j^* &= c_j - \frac{c_l}{a_{kl}} a_{kj}, \quad j \neq k, \\ c_k^* &= -\frac{c_l}{a_{kl}}. \end{aligned}$$

We can now answer Exercise 2.18. The question is: can the variable x_k, which is now non-basic, enter the basis in the new dictionary? Well, for x_k to enter the basis, we would require that $c_k^* > 0$. However, by our previous assumptions,

$$c_k^* = -\frac{c_l}{a_{kl}} < 0,$$

and so the answer is no.

Exercise 2.19

The initial dictionary is

$$\begin{aligned} \eta &= \sum_{j=1}^{n} p_j x_j \\ w_1 &= 1 - x_1 \\ &\vdots \\ w_n &= 1 - x_n \\ w_{n+1} &= \beta - \sum_{j=1}^{n} q_j x_j. \end{aligned}$$

Let's put x_n into the basis. Since β is small, w_{n+1} leaves the basis. Then, since

$$w_{n+1} = \beta - \sum_{j=1}^{n-1} q_j x_j - q_n x_n,$$

we obtain

$$x_n = \frac{1}{q_n}\left(\beta - \sum_{j=1}^{n-1} q_j x_j - w_{n+1}\right),$$

and

$$\begin{aligned} \eta &= \sum_{j=1}^{n-1} p_j x_j + p_n x_n \\ &= \sum_{j=1}^{n-1} p_j x_j + \frac{p_n}{q_n}\left(\beta - \sum_{j=1}^{n-1} q_j x_j - w_{n+1}\right) \\ &= \frac{p_n}{q_n}\beta + \sum_{j=1}^{n-1}\left(p_j - \frac{p_n}{q_n} q_j\right) x_j - \frac{p_n}{q_n} w_{n+1}. \end{aligned}$$

Since

$$p_j - \frac{p_n}{q_n} q_j < 0, \quad j = 1, \ldots, n-1,$$

this is an optimal dictionary, and the optimal solution is

$$x_1 = \cdots = x_{n-1} = 0, \quad x_n = \frac{\beta}{q_n}, \quad \eta = \frac{p_n}{q_n}\beta.$$

Exercise 3.1

This is the same LP problem as discussed in the lecture notes: 'Chapter 3: degeneracy'. The initial dictionary, Dictionary 0, is

$$\begin{array}{rcrcrcrcrcr}
\eta & = & & + & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \\
\hline
w_1 & = & 0 & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
w_2 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
w_3 & = & 1 & - & x_1 & & & & & &
\end{array}$$

Recall that in the notes we used the largest coefficient rule for the entering variable and the smallest index rule for the leaving variable. With these rules we saw that this example cycles: Dictionary 6 = Dictionary 0.

Here we prevent cycling using the lexicographic rule. We introduce

$$0 < \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll \text{all other data.}$$

Then the initial dictionary is

$$\begin{array}{rcrcrcrcrcr}
\eta & = & & + & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \\
\hline
w_1 & = & \epsilon_1 & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
w_2 & = & \epsilon_2 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
w_3 & = & 1+\epsilon_3 & - & x_1 & & & & & &
\end{array}$$

Then x_1 enters and w_2 leaves:

$$\begin{array}{rcrcrcrcrcr}
\eta & = & 20\epsilon_2 & - & 20w_2 & - & 27x_2 & + & x_3 & - & 44x_4 \\
\hline
w_1 & = & \epsilon_1 - \epsilon_2 & + & w_2 & + & 4x_2 & + & 2x_3 & - & 8x_4 \\
x_1 & = & 2\epsilon_2 & - & 2w_2 & + & 3x_2 & + & x_3 & - & 2x_4 \\
w_3 & = & 1 - 2\epsilon_2 + \epsilon_3 & + & 2w_2 & - & 3x_2 & - & x_3 & + & 2x_4
\end{array}$$

Now x_3 enters and w_3 leaves:

$$\begin{array}{rcrcrcrcrcr}
\eta & = & 1 - 18\epsilon_2 + \epsilon_3 & - & 18w_2 & - & 30x_2 & - & w_3 & - & 42x_4 \\
\hline
w_1 & = & 2 + \epsilon_1 - 5\epsilon_2 + 2\epsilon_3 & + & 5w_2 & - & 2x_2 & - & 2w_3 & - & 4x_4 \\
x_1 & = & 1 + \epsilon_3 & & & & & - & w_3 & & \\
x_3 & = & 1 - 2\epsilon_2 + \epsilon_3 & + & 2w_2 & - & 3x_2 & - & w_3 & + & 2x_4
\end{array}$$

This is optimal. We now delete the ϵ's and we get

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 0, \quad \eta = 1.$$

Exercise 3.2

This is again the same LP problem. Now we use Bland's rule to prevent cycling. Bland's rule is: use the smallest index rule for both entering and leaving variables. Using Bland's rule, we find that the first five iterations for this example are the same as in the lectures notes (Chapter 3) and Dictionary 5 is:

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 0 & - & 21x_3 & + & 24w_2 & + & 22x_1 & - & 93x_2 \\
\hline
w_1 & = & 0 & - & 2x_3 & + & 9w_2 & + & 4x_1 & - & 8x_2 \\
x_4 & = & 0 & + & 0.5x_3 & - & w_2 & - & 0.5x_1 & + & 1.5x_2 \\
w_3 & = & 1 & & & & & - & x_1 & &
\end{array}
$$

But now we get a change due to Bland's rule. Although w_2 has the largest positive coefficient, Bland's rule chooses x_1 as the entering variable since it has the smallest index among the non-basic variables with positive coefficients (and we are identifying $w_1 = x_5$, $w_2 = x_6$, $w_3 = x_7$). So x_1 enters and x_4 leaves:

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 0 & + & x_3 & - & 20w_2 & - & 44x_4 & - & 27x_2 \\
\hline
w_1 & = & 0 & + & 2x_3 & + & w_2 & - & 8x_4 & + & 4x_2 \\
x_1 & = & 0 & + & x_3 & - & 2w_2 & - & 2x_4 & + & 3x_2 \\
w_3 & = & 1 & - & x_3 & + & 2w_2 & + & 2x_4 & - & 3x_2
\end{array}
$$

Now x_3 enters and w_3 leaves:

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 1 & - & w_3 & - & 18w_2 & - & 42x_4 & - & 30x_2 \\
\hline
w_1 & = & 2 & - & 2w_3 & + & 5w_2 & - & 4x_4 & - & 2x_2 \\
x_1 & = & 1 & - & w_3 & & & & & & \\
x_3 & = & 1 & - & w_3 & + & 2w_2 & + & 2x_4 & - & 3x_2
\end{array}
$$

This is optimal and the same as in Ex. 3.1.

Exercise 3.4

In this LP problem all the hyperplanes defining the boundary of the feasible region pass through the origin. So the feasible region is either just the origin or an unbounded cone with apex at the origin. So depending on the objective function, the problem is either unbounded or has the origin as an optimal solution.

To prove this, suppose that $(0,\dots,0)$ is not optimal. Then there exists some point $x = (x_1,\dots,x_n) \neq 0$ such that

$$\eta(x) = \sum_{j=1}^{n} c_j x_j > 0$$

and

$$\sum_{j=1}^{n} a_{ij} x_j \leq 0, \quad i = 1,\dots,m.$$

Then consider the point $x' = \lambda x$ for some $\lambda > 0$. We have

$$\eta(x') = \eta(\lambda x) = \lambda \eta(x)$$

and

$$\sum_{j=1}^{n} a_{ij} x'_j = \lambda \sum_{j=1}^{n} a_{ij} x_j \leq 0, \quad i = 1,\dots,m.$$

Therefore, all points $x' = \lambda x$, $\lambda > 0$, are in the feasible region and $\eta(x') \to \infty$ as $\lambda \to \infty$. So the problem is unbounded.