Answers to Exercises, Week 5, MAT3100, V20

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Exercises in Week 5 are: Ex. 4.1, 4.2, 4.4, 4.6, 4.7.

Exercise 4.1

Largest coefficient rule:

$$
\begin{array}{rcrcrcr}
\eta & = & & + & 4x_1 & + & 5\mathbf{x_2} \\
\hline
w_1 & = & 9 & - & 2x_1 & - & 2x_2 \\
w_2 & = & 4 & - & x_1 & & \\
\mathbf{w_3} & = & 3 & & & - & x_2
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\eta & = & 15 & + & 4\mathbf{x_1} & - & 5w_3 \\
\hline
\mathbf{w_1} & = & 3 & - & 2x_1 & + & 2w_3 \\
w_2 & = & 4 & - & x_1 & & \\
x_2 & = & 3 & & & - & w_3
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\eta & = & 21 & - & 2w_1 & - & w_3 \\
\hline
x_1 & = & 1.5 & - & 0.5x_1 & + & w_3 \\
w_2 & = & 2.5 & + & 0.5w_1 & - & w_3 \\
x_2 & = & 3 & & & - & w_3
\end{array}
$$

Two iterations required.

Smallest index rule:

$$
\begin{array}{rcrcrcr}
\eta & = & & + & 4\mathbf{x_1} & + & 5x_2 \\
\hline
w_1 & = & 9 & - & 2x_1 & - & 2x_2 \\
\mathbf{w_2} & = & 4 & - & x_1 & & \\
w_3 & = & 3 & & & - & x_2
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\eta & = & 16 & - & 4w_2 & + & 5\mathbf{x_2} \\
\hline
\mathbf{w_1} & = & 1 & + & 2x_1 & - & 2x_2 \\
x_1 & = & 4 & - & w_2 & & \\
x_2 & = & 3 & & & - & x_2
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\eta & = & 18.5 & + & \mathbf{w_2} & - & 2.5w_1 \\
\hline
x_2 & = & 0.5 & + & w_2 & - & 0.5w_1 \\
x_1 & = & 4 & - & w_2 & & \\
\mathbf{w_3} & = & 2.5 & - & w_2 & + & 0.5w_1
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\eta & = & 21 & - & w_3 & - & 2w_1 \\
\hline
x_2 & = & 3 & - & w_3 & & \\
x_1 & = & 1.5 & + & w_3 & - & 0.5w_1 \\
w_2 & = & 2.5 & - & w_3 & + & 0.5w_1
\end{array}
$$

Three iterations required.

Exercise 4.2

Largest coefficient rule:

$$
\begin{array}{rcrcrcr}
\eta & = & & + & 2\mathbf{x_1} & + & x_2 \\
\hline
\mathbf{w_1} & = & 3 & - & 3x_1 & - & x_2
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\eta & = & 2 & - & (2/3)w_1 & + & (1/3)\mathbf{x_2} \\
\hline
\mathbf{x_1} & = & 1 & - & (1/3)w_1 & - & (1/3)x_2
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\eta & = & 3 & - & w_1 & - & x_1 \\
\hline
x_2 & = & 3 & - & w_1 & - & 3x_1
\end{array}
$$

Two iterations required. The same for the smallest index rule.

Exercise 4.4

The Klee-Minty cube for $n = 3$ was solved in the notes: ‘Lecture 4’.

Exercise 4.6

Consider the dictionary

$$
\begin{aligned}
\eta &= -\sum_{j=1}^{n} \epsilon_j 10^{n-j} \left(\frac{1}{2} b_j - x_j \right) \\
w_i &= \sum_{j=1}^{i-1} \epsilon_i \epsilon_j 10^{i-j} (b_j - 2x_j) + (b_i - x_i), \quad i = 1, \ldots, n,
\end{aligned}
$$

where ϵ_i is ± 1, $i = 1, \dots, n$, and

$$1 = b_1 \ll b_2 \ll b_3 \ll \cdots \ll b_n.$$

Suppose that we choose k and put x_k into the basis and remove w_k. Then we need to compute the new dictionary. From the equation for w_k we get

$$x_k = \sum_{j=1}^{k-1} \epsilon_k \epsilon_j 10^{k-j} (b_j - 2x_j) + (b_k - w_k).$$

We then substitute this into the other rows of the dictionary. For $i = 1, \dots, k-1$, there is no change to w_i. For $i = k+1, \dots, n$, we find

$$\begin{aligned}
w_i &= \sum_{\substack{j=1 \\ j \neq k}}^{i-1} \epsilon_i \epsilon_j 10^{i-j} (b_j - 2x_j) + (b_i - x_i) + \epsilon_i \epsilon_k 10^{i-k} (b_k - 2x_k) \\
&= \sum_{\substack{j=1 \\ j \neq k}}^{i-1} \epsilon_i \epsilon_j 10^{i-j} (b_j - 2x_j) + (b_i - x_i) \\
&\quad + \epsilon_i \epsilon_k 10^{i-k} \left(b_k - 2 \sum_{j=1}^{k-1} \epsilon_k \epsilon_j 10^{k-j} (b_j - 2x_j) - 2(b_k - w_k) \right) \\
&= - \sum_{j=1}^{k-1} \epsilon_i \epsilon_j 10^{i-j} (b_j - 2x_j) - \epsilon_i \epsilon_k 10^{i-k} (b_k - 2w_k) \\
&\quad + \sum_{j=k+1}^{i-1} \epsilon_i \epsilon_j 10^{i-j} (b_j - 2x_j) + (b_i - x_i),
\end{aligned}$$

where we used the fact that $1-2\epsilon_k^2=-1$. Similarly, for η:

$$\begin{aligned}
\eta &= -\sum_{\substack{j=1\\ j\neq k}}^{n} \epsilon_j 10^{n-j}\left(\frac{1}{2}b_j - x_j\right) - \epsilon_k 10^{n-k}\left(\frac{1}{2}b_k - x_k\right) \\
&= -\sum_{\substack{j=1\\ j\neq k}}^{n} \epsilon_j 10^{n-j}\left(\frac{1}{2}b_j - x_j\right) \\
&\quad - \epsilon_k 10^{n-k}\left(\frac{1}{2}b_k - \sum_{j=1}^{k-1}\epsilon_k\epsilon_j 10^{k-j}(b_j-2x_j) - (b_k - w_k)\right) \\
&= \sum_{j=1}^{k-1} \epsilon_j 10^{n-j}\left(\frac{1}{2}b_j - x_j\right) + \epsilon_k 10^{n-k}\left(\frac{1}{2}b_k - w_k\right) \\
&\quad - \sum_{j=k+1}^{n} \epsilon_j 10^{n-j}\left(\frac{1}{2}b_j - x_j\right).
\end{aligned}$$

If we now define

$$(\epsilon_1^*,\ldots,\epsilon_n^*) = (-\epsilon_1,\ldots,-\epsilon_k,\epsilon_{k+1},\ldots,\epsilon_n),$$

and

$$\begin{aligned}
x_k^* &= w_k, \quad w_k^* = x_k, \\
x_i^* &= x_i, \quad w_i^* = w_i, \quad i\neq k,
\end{aligned}$$

the new dictionary can be written as

$$\begin{aligned}
\eta &= -\sum_{j=1}^{n} \epsilon_j^* 10^{n-j}\left(\frac{1}{2}b_j - x_j^*\right) \\
w_i^* &= \sum_{j=1}^{i-1} \epsilon_i^*\epsilon_j^* 10^{i-j}(b_j - 2x_j^*) + (b_i - x_i^*), \quad i=1,\ldots,n,
\end{aligned}$$

which is the same as the old one except that ϵ_i, x_i, and w_i have been replaced by ϵ_i^*, x_i^*, and w_i^*.

Exercise 4.7

Recall from the lecture notes that the modified Klee-Minty problem is

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} 10^{n-j} x_j - (1/2) \sum_{j=1}^{i-1} 10^{n-j} b_j & \\ \text{subject to} & 2\sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \quad \leq \sum_{j=1}^{i-1} 10^{i-j} b_i, & i \leq n, \\ & x_j \quad \geq 0, & j \leq n, \end{array}$$

and

$$1 = b_1 \ll b_2 \ll b_3 \ll \cdots \ll b_n.$$

So the initial dictionary is

$$\begin{aligned} \eta &= -\sum_{j=1}^{n} 10^{n-j} \left(\frac{1}{2} b_j - x_j \right) \\ w_i &= \sum_{j=1}^{i-1} 10^{i-j} (b_j - 2x_j) + (b_j - x_j). \end{aligned}$$

Now, for each of the 2^n choices of $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ where $\epsilon_i = \pm 1$ for each $i = 1, \ldots, n$, let D_ϵ be the dictionary

$$\begin{aligned} \eta &= -\sum_{j=1}^{n} \epsilon_j 10^{n-j} \left(\frac{1}{2} b_j - \overline{x}_j \right) \\ \overline{w}_i &= \sum_{j=1}^{i-1} \epsilon_i \epsilon_j 10^{i-j} (b_j - 2\overline{x}_j) + (b_i - \overline{x}_i), \quad i = 1, \ldots, n, \end{aligned}$$

where, for each $i = 1, \ldots, n$,

$$(\overline{x}_i, \overline{w}_i) = \begin{cases} (x_i, w_i) & \text{if } \epsilon_i \epsilon_{i+1} = 1; \\ (w_i, x_i) & \text{if } \epsilon_i \epsilon_{i+1} = -1, \end{cases} \tag{1}$$

and we have defined $\epsilon_{n+1} := 1$.

To show that the simplex method takes $2^n - 1$ iterations using the largest coefficient rule, we will show the the method passes through every dictionary of the form D_ϵ. To prove this, observe that the initial dictionary is $D_{(1,1,\dots,1)}$. Next suppose that D_ϵ is the current dictionary. Looking at η, by the largest coefficient rule, we choose $\overline{x}_k$ to enter the basis, where k is the smallest index in $\{1,\dots,n\}$ such that $\epsilon_k = 1$. Then, $\overline{w}_k$ leaves the basis, and by the calculation in Exercise 4.6, and using (1), the new dictionary is D_{ϵ^*}, where

$$(\epsilon_1^*,\dots,\epsilon_n^*) = (-\epsilon_1,\dots,-\epsilon_k,\epsilon_{k+1},\dots,\epsilon_n). \tag{2}$$

If on the other hand $\epsilon = (-1,\dots,-1)$, dictionary D_ϵ is optimal.

Thus it remains to show that every ϵ in the set

$$E_n = \{\epsilon = (\epsilon_1,\dots,\epsilon_n) : \epsilon_i = \pm 1, i = 1,\dots,n\}$$

is visited precisely once, starting with $\epsilon = (1,\dots,1)$ and ending with $\epsilon = (-1,\dots,-1)$. This is clearly true in the case $n = 1$ since we pass from $\epsilon = (1)$ to $\epsilon = (-1)$ in one iteration. By induction on n, let us suppose that this property holds with n replaced by $n-1$. Then we can assume that we iterate from $\epsilon = (1,\dots,1,1)$ to $(-1,\dots,-1,1)$ in $2^{n-1}-1$ steps, passing through every ϵ of the form $\epsilon = (\delta,1)$ for $\delta \in E_{n-1}$. The next step takes us to $(1,\dots,1,-1)$. And, then, again by the induction hypothesis, we iterate from $(1,\dots,1,-1)$ to $(-1,\dots,-1,-1)$ passing through every ϵ of the form $\epsilon = (\delta,-1)$ for $\delta \in E_{n-1}$. Thus this property does indeed hold for all n.

Note also that in the first dictionary, $D_{(1,\dots,1)}$, we have

$$x_1 = \cdots = x_n = 0,$$

and in the last dictionary, $D_{(-1,\dots,-1)}$, we have

$$x_1 = \cdots = x_{n-1} = 0, \quad x_n \neq 0.$$

If we simplify the feasible region to the unit cube, the case $n = 3$ is as follows:

ϵ	$x = (x_1, x_2, x_3)$
$(1,1,1)$	$(0,0,0)$
$(-1,1,1)$	$(1,0,0)$
$(1,-1,1)$	$(1,1,0)$
$(-1,-1,1)$	$(0,1,0)$
$(1,1,-1)$	$(0,1,1)$
$(-1,1,-1)$	$(1,1,1)$
$(1,-1,-1)$	$(1,0,1)$
$(-1,-1,-1)$	$(0,0,1)$