UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	INF-MAT 3370 — Linear optimization
Day of examination:	May 31., 2011
Examination hours:	09.00 – 13.00

This problem set consists of 3 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same weight.

Problem 1

Consider the LP problem:

$$
\begin{array}{lrcrcrcr}
\text{max} & x_1 & - & 2x_2 & + & x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & 2x_2 & + & x_3 & \leq & 12 \\
 & 2x_1 & + & x_2 & - & x_3 & \leq & 6 \\
 & -x_1 & + & 3x_2 & & & \leq & 9 \\
 & & & x_1, x_2, x_3 \geq 0. & & & &
\end{array}
\tag{1}
$$

1a

Solve problem (1) using the simplex algorithm with initial point $x = (0, 0, 0)$. Find an optimal solution, including values on the slack variables, and the optimal value.

1b

Find the dual problem of (1). Moreover, find an optimal solution of the dual, including dual slacks, preferable without any computations.

Let $a \leq 0$ be a parameter (number) and define the function $f_a : \mathbb{R}^3 \to \mathbb{R}$ by $f_a(x_1, x_2, x_3) = x_1 + ax_2 + x_3$. Let (P) denote the problem obtained from the LP problem (1) by replacing the objective function by $f_a(x_1, x_2, x_3)$, but using the same constraints.

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1c

Find an x^* which is optimal in (P) for all $a \leq 0$, and show that it is optimal.

Problem 2

Consider the LP problem:

$$\begin{array}{lrcrcrcr}
\text{max} & -x_1 & - & x_2 & - & x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & 2x_2 & + & x_3 & \leq & 12 \\
& 2x_1 & + & x_2 & - & x_3 & \leq & -6 \\
& & & x_1, x_2, x_3 \geq 0. & & & &
\end{array} \tag{2}$$

2a

Use the dual simplex method to find a feasible solution in (2). (If you don't remember this method, you may also get some (but not full) score using another method.) Is there a basic feasible solution in (2) where x_3 is not a basic variable (i.e., x_3 is not in the basis)? Give reasons for your answer.

Problem 3

Let $a \in \mathbb{R}^n$ where $\|a\| = \sqrt{a^T a} = 1$, and let $b_1, b_2 \in \mathbb{R}$ with $b_1 < b_2$. Define $H_1 = \{x \in \mathbb{R}^n : a^T x = b_1\}$ and $H_2 = \{x \in \mathbb{R}^n : a^T x = b_2\}$.

3a

Determine the convex hull of $H_1 \cup H_2$ (with proof).

Problem 4

Consider the LP problem

$$\begin{array}{ll}
\text{max} & c^T x \\
\text{subject to} & \\
& Ax = b \\
& O \leq x \leq h
\end{array} \tag{3}$$

Here $c, h \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and the $m \times n$ matrix A are given, and O denotes the zero vector.

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4a

Find the dual of problem (2).

Consider the linear system

$$\begin{array}{l} \sum_{j=1}^{n} x_j \leq 1 \\ O \leq x_j \leq h_j \quad (j \leq n) \end{array} \tag{4}$$

where each h_j is (strictly) positive. (Note: there are only inequalities.)

4b

Use Fourier-Motzkin elimination to eliminate x_1 in (4). Then go on and eliminate x_2, x_3 etc.; explain how x_k depends on $x_{k+1}, \ldots, x_n$ in a general solution of (4).

Problem 5

Consider the following minimum cost network flow problem. Define the directed graph $D = (V, E)$ where $V = \{v_1, v_2, \ldots, v_5\}$ (the nodes) and E (the edges=arcs) consists of (v_i, v_{i+1}) for $1 \leq i \leq 4$ and the edge (v_2, v_4). So D has 5 edges. Define the supply vector b by $b_{v_1} = 1$, $b_{v_5} = -1$ and $b_{v_i} = 0$ for $i = 2, 3, 4$. Finally, define the cost $c_e = 1$ for every edge e.

5a

Draw the graph. Find all spanning trees in D (for each, give its edges). Let T_1 be the spanning tree which does not contain (v_3, v_4), and compute the corresponding tree solution x^*.

5b

Compute the dual solution (y, z) corresponding to the spanning tree T_1 above: let here $y_{v_1} = 0$ (so v_1 is the root). Explain why x^* above is an optimal solution of the minimum cost network flow problem.

Let A be the node-arc (node-edge) incidence matrix of the graph D above.

5c

Let r be the maximum rank of a square submatrix of A. Find r and a submatrix B of A such that B has rank r. Explain your answer with reference to general theory.

Good luck!