

# UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in        INF-MAT 3370 — Linear optimization

Day of examination:    May 31., 2011

Examination hours:    09.00–13.00

This problem set consists of 3 pages.

Appendices:            None

Permitted aids:        None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same weight.

## Problem 1

Consider the LP problem:

$$\begin{array}{ll} \max & x_1 - 2x_2 + x_3 \\ \text{subject to} & \\ & x_1 + 2x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 - x_3 \leq 6 \\ & -x_1 + 3x_2 \leq 9 \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{1}$$

### 1a

Solve problem (1) using the simplex algorithm with initial point  $x = (0, 0, 0)$ . Find an optimal solution, including values on the slack variables, and the optimal value.

### 1b

Find the dual problem of (1). Moreover, find an optimal solution of the dual, including dual slacks, preferable without any computations.

Let  $a \leq 0$  be a parameter (number) and define the function  $f_a : \mathbb{R}^3 \rightarrow \mathbb{R}$  by  $f_a(x_1, x_2, x_3) = x_1 + ax_2 + x_3$ . Let (P) denote the problem obtained from the LP problem (1) by replacing the objective function by  $f_a(x_1, x_2, x_3)$ , but using the same constraints.

*(Continued on page 2.)*

**1c**

Find an  $x^*$  which is optimal in (P) for all  $a \leq 0$ , and show that it is optimal.

**Problem 2**

Consider the LP problem:

$$\begin{aligned} \max \quad & -x_1 - x_2 - x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 - x_3 \leq -6 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{2}$$

**2a**

Use the dual simplex method to find a feasible solution in (2). (If you don't remember this method, you may also get some (but not full) score using another method.) Is there a basic feasible solution in (2) where  $x_3$  is not a basic variable (i.e.,  $x_3$  is not in the basis)? Give reasons for your answer.

**Problem 3**

Let  $a \in \mathbb{R}^n$  where  $\|a\| = \sqrt{a^T a} = 1$ , and let  $b_1, b_2 \in \mathbb{R}$  with  $b_1 < b_2$ . Define  $H_1 = \{x \in \mathbb{R}^n : a^T x = b_1\}$  and  $H_2 = \{x \in \mathbb{R}^n : a^T x = b_2\}$ .

**3a**

Determine the convex hull of  $H_1 \cup H_2$  (with proof).

**Problem 4**

Consider the LP problem

$$\begin{aligned} \max \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & 0 \leq x \leq h \end{aligned} \tag{3}$$

Here  $c, h \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$  and the  $m \times n$  matrix  $A$  are given, and  $0$  denotes the zero vector.

(Continued on page 3.)

**4a**

Find the dual of problem (3).

Consider the linear system

$$\begin{aligned} \sum_{j=1}^n x_j &\leq 1 \\ 0 &\leq x_j \leq h_j \quad (j \leq n) \end{aligned} \quad (4)$$

where each  $h_j$  is (strictly) positive. (Note: there are only inequalities.)

**4b**

Use Fourier-Motzkin elimination to eliminate  $x_1$  in (4). Then go on and eliminate  $x_2, x_3$  etc.; explain how  $x_k$  depends on  $x_{k+1}, \dots, x_n$  in a general solution of (4).

**Problem 5**

Consider the following minimum cost network flow problem. Define the directed graph  $D = (V, E)$  where  $V = \{v_1, v_2, \dots, v_5\}$  (the nodes) and  $E$  (the edges=arcs) consists of  $(v_i, v_{i+1})$  for  $1 \leq i \leq 4$  and the edge  $(v_2, v_4)$ . So  $D$  has 5 edges. Define the supply vector  $b$  by  $b_{v_1} = 1, b_{v_5} = -1$  and  $b_{v_i} = 0$  for  $i = 2, 3, 4$ . Finally, define the cost  $c_e = 1$  for every edge  $e$ .

**5a**

Draw the graph. Find all spanning trees in  $D$  (for each, give its edges). Let  $T_1$  be the spanning tree which does not contain  $(v_3, v_4)$ , and compute the corresponding tree solution  $x^*$ .

**5b**

Compute the dual solution  $(y, z)$  corresponding to the spanning tree  $T_1$  above: let here  $y_{v_1} = 0$  (so  $v_1$  is the root). Explain why  $x^*$  above is an optimal solution of the minimum cost network flow problem.

Let  $A$  be the node-arc (node-edge) incidence matrix of the graph  $D$  above.

**5c**

Let  $r$  be the maximum rank of a square submatrix of  $A$ . Find  $r$  and a submatrix  $B$  of  $A$  such that  $B$  has rank  $r$ . Explain your answer with reference to general theory.

*Good luck!*