UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3100 — Linear Optimization

Day of examination: Monday, June 1st, 2015

Examination hours: 09.00 – 13.00

This problem set consists of 8 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider the LP problem

$$\begin{array}{lrcrcrcl} \text{maximize} & -7x_1 & & & + & 2x_3 & & \\ \text{subject to} & & & & & & & \\ & & - & 3x_2 & + & 4x_3 & \leq & 1, \\ & x_1 & - & x_2 & & & \leq & 2, \\ & -3x_1 & & & + & x_3 & \leq & 0, \\ & & & & & x_1, x_2, x_3 & \geq & 0. \end{array} \tag{1}$$

1a

Use the simplex algorithm to find an optimal solution and the accompanying optimal value.

<u>Answer:</u> Initial dictionary

$$\begin{aligned} \max \eta &= -7x_1 + 2x_3, \\ w_1 &= 1 + 3x_2 - 4x_3, \\ w_2 &= 2 - x_1 + x_2, \\ w_3 &= 3x_1 - x_3. \end{aligned}$$

We perform a pivot step with x_3 into the basis and w_3 out of the basis (so $x_3 = 3x_1 - w_3$), resulting in the dictionary

$$\begin{aligned} \max \eta &= -x_1 - 2w_3, \\ w_1 &= 1 - 12x_1 + 3x_2 + 4w_3, \\ w_2 &= 2 - x_1 + x_2, \\ x_3 &= 3x_1 - w_3. \end{aligned} \tag{2}$$

An optimal solution is $x_1 = x_2 = w_3 = 0$ and $w_1 = 1$, $w_2 = 2$, $x_3 = 0$. The optimal value is $\eta = 0$.

(Continued on page 2.)

1b

Determine all the optimal solutions of (1).

Answer: In view of (2), there is a whole family of optimal solutions corresponding to the objective value $\eta = 0$, namely $x_1 = 0$, $x_2 = t$ for any number $t \geq 0$, $w_3 = 0$ and $w_1 = 1 + 3t$, $w_2 = 2 + t$, $x_3 = 0$.

1c

Consider the LP problem

$$\begin{array}{lrcrcrcr} \text{minimize} & y_1 & + & 2y_2 & & & & \\ \text{subject to} & & & & & & & \\ & & & y_2 & - & 3y_3 & \geq & -7, \\ & -3y_1 & - & y_2 & & & \geq & 0, \\ & 4y_1 & & & + & y_3 & \geq & 2, \\ & & & & & y_1, y_2, y_3 & \geq & 0. \end{array} \tag{3}$$

Use duality and your findings in (1a) to obtain an optimal solution of (3) and the accompanying optimal value.

Answer: We note that (3) is the dual of (1), to which we can apply the simplex algortihm. The dual variables y_1, y_2, y_3 corresponds to the primal slack variables w_1, w_2, w_3, while the dual slack variables z_1, z_2, z_3 corresponds to the primal variables x_1, x_2, x_3. For the primal dictionary (2), the basic variables are w_1, w_2, x_3, while the nonbasic variables are x_1, x_2, w_3. For the dual dictionary, the basic variables becomes z_1, z_2, y_3, while the nonbasic variables becomes y_1, y_2, z_3. In view of the "negative-transpose property", the dual dictionary reads

$$\begin{aligned} &\min -\xi = -y_1 - 2y_2, \\ &z_1 = 1 + 12y_1 + y_2 - 3z_3, \\ &z_2 = -3y_1 - y_2, \\ &y_3 = 2 - 4y_1 + z_3. \end{aligned} \tag{4}$$

Hence, the optimal solution is $y_1 = 0$, $y_2 = 0$, $z_3 = 0$ and $z_1 = 1$, $z_2 = 0$, $y_3 = 2$. The optimal value is 0.

Problem 2

2a

Consider the (primal) LP problem

$$\begin{aligned} &\max c^T x, \\ &\text{subject to } Ax \leq b,\ x \geq 0, \end{aligned} \tag{5}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $x, c \in \mathbb{R}^n$.

State the weak and strong duality theorems. Moreover, prove the weak duality theorem.

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Answer: The dual of (5) is

$$\begin{aligned}&\min b^T y,\\&\text{subject to } A^T y \geq c,\ y \geq 0.\end{aligned} \tag{6}$$

If $x \in \mathbb{R}^n$ is primal feasible and $y \in \mathbb{R}^m$ is dual feasible, then the weak duality theorem states that

$$c^T x \leq b^T y.$$

If the primal problem (5) has an optimal solution $x^\star$, then the dual (6) also has an optimal solution $y^\star$, such that

$$c^T x^\star = b^T y^\star.$$

The proof of the weak duality theorem proceeds as follows:

$$\begin{aligned}\sum_j c_j x_j &\leq \sum_j \left(\sum_i y_i a_{ij}\right) x_j\\&= \sum_i \left(\sum_j a_{ij} x_j\right) y_i\\&\leq \sum_i b_i y_i,\end{aligned}$$

where we have used that $x_j \geq 0$ and $c_j \leq \sum_i y_i a_{ij}$, and moreover that $y_i \geq 0$ and $b_i \geq \sum_j a_{ij}$.

2b

Let $x = (x_1, x_2, \ldots, x_n)^T$ be primal feasible and $y = (y_1, y_2, \ldots, y_m)^T$ dual feasible. Denote by $(w_1, w_2, \ldots, w_m)$ the corresponding primal slack variables and $(z_1, z_2, \ldots, z_n)$ the corresponding dual slack variables.

Suppose x is optimal for the primal problem and y is optimal for the dual problem. State and prove the complementary slackness equations.

Answer: The complementary slackness equations read

$$x_j z_j = 0, \qquad j = 1, \ldots, n$$

and

$$w_i y_i = 0, \qquad i = 1, \ldots, m.$$

By the proof of the weak duality theorem,

$$\begin{aligned}\sum_j c_j x_j &\leq \sum_j \left(\sum_i y_i a_{ij}\right) x_j\\&= \sum_i \left(\sum_j a_{ij} x_j\right) y_i\\&\leq \sum_i b_i y_i.\end{aligned}$$

(Continued on page 4.)

The first inequality is actually an equality (due to optimality), and therefore

$$x_j = 0 \quad \text{or} \quad c_j = \sum_i y_i a_{ij}, \qquad j = 1, \ldots, n.$$

Since $z_j = \sum_i y_i a_{ij} - c_j$, the last part takes the form $z_j = 0$. Thus, optimality implies $x_j z_j = 0$, for all j.

The second inequality is also an equality (due to optimality), and so

$$\sum_j a_{ij} x_j = b_i \quad \text{or} \quad y_i = 0, \qquad i = 1, \ldots, m,$$

and, since $w_i = b_i - \sum_j a_{ij} x_j$, the first part takes the form $w_i = 0$. Thus, optimality implies $w_i y_i = 0$, for all i.

2c

$$\begin{array}{llcll} \text{maximize} & 3x_1 & + & 2x_2 & \\ \text{subject to} & & & & \\ & 2x_1 & + & x_2 & \leq 4, \\ & 2x_1 & + & 3x_2 & \leq 6, \\ & & & x_1, x_2 & \geq 0. \end{array} \tag{7}$$

Show that $x^\star = (3/2, 1)$ is primal feasible and $y^\star = (5/4, 1/4)$ is dual feasible. Moreover, show that $x^\star$ is in fact an optimal solution of (7).

<u>Answer:</u> The dual of (7) is

$$\begin{array}{ll} \min & 4y_1 + 6y_2, \\ \text{subject to} & \\ & 2y_1 + 2y_2 \geq 3, \\ & y_1 + 3y_2 \geq 2, \\ & y_1, y_2 \geq 0. \end{array} \tag{8}$$

Straightforward computations reveal that $x^\star = (3/2, 1)$ and $y^\star = (5/4, 1/4)$ satisfy their respective constraints:

$$2 * 3/2 + 1 = 4, \qquad 2 * 3/2 + 3 * 1 = 6$$

and

$$2 * 5/4 + 2 * 1/4 = 12/4 = 3, \qquad 5/4 + 3 * 1/4 = 8/4 = 2.$$

Moreover, the objective function values $\eta^\star, \xi^\star$ of the primal and dual problems coincide:

$$\eta^\star := 3 * 3/2 + 2 * 1 = 13/2$$

and

$$\xi^\star := 4 * 5/4 + 6 * 1/4 = 26/4 = 13/2.$$

We then conclude by the weak duality theorem that $x^\star$ and $y^\star$ are optimal in their respective problems.

(Continued on page 5.)

2d

Consider the LP problem

$$\begin{aligned} &\max c^T x, \\ &\text{subject to } Ax = b,\ x \geq 0, \end{aligned} \tag{9}$$

where the linear constraints are equalities. Show that the dual of (9) is

$$\begin{aligned} &\min b^T y, \\ &\text{subject to } A^T y \geq c. \end{aligned} \tag{10}$$

<u>Answer:</u> First, we write the equality constraint as inequality constraints:

$$\begin{aligned} &\max c^T x, \\ &\text{subject to } Ax \leq b,\ -Ax \leq -b,\ x \geq 0, \end{aligned}$$

or, in terms of partitioned matrices,

$$\begin{aligned} &\max c^T x, \\ &\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix}, \\ &x \geq 0. \end{aligned}$$

Given this LP problem in standard form, we can write down its dual, using two duality variables (vectors) y^+ and y^-:

$$\begin{aligned} &\min b^T y^+ - b^T y^-, \\ &\text{subject to } A^T y^+ - A^T y^- \geq c, \quad y^+, y^- \geq 0. \end{aligned}$$

If we set $y := y^+ - y^-$ (y is not necessarily nonnegative), the dual problem of (9) becomes

$$\min b^T y, \qquad \text{subject to } A^T y \geq c.$$

Problem 3

3a

Consider the LP problem

$$\begin{aligned} &\max \sum_{j=1}^{n} c_j x_j, \\ &\text{subject to} \\ &\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m, \\ &x_j \geq 0, \quad j = 1, \ldots, n, \end{aligned} \tag{11}$$

where c_j, a_{ij}, b_i are given numbers.

(Continued on page 6.)

Introduce slack variables and identify vectors x, c, b and a matrix A such that (11) can be written as

$$\begin{aligned} &\max c^T x, \\ &\text{subject to } Ax = b,\ x \geq 0. \end{aligned} \tag{12}$$

Answer: Introduce the slack variables:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \dots, m,$$

and write

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} \mathbb{R}^{m+n}.$$

Take $A = \begin{bmatrix} \bar{A} & I \end{bmatrix}$, where I is the $m \times m$ identity matrix and

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Moreover, take

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{c} \\ 0 \end{pmatrix} \in \mathbb{R}^{m+n}$$

and

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m.$$

Then (11) can be written in the form (12).

3b

In the simplex algorithm denote by $\mathcal{B}$ the set of indices corresponding to the basic variables, and by $\mathcal{N}$ the remaining nonbasic indices.

Let B denote an invertible $m \times m$ matrix whose columns consist of the m columns of A associated with the basic variables. Similarly, denote by N an $m \times n$ matrix whose columns are the n nonbasic columns of A.

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Assume that A can be partitioned as

$$A = \begin{bmatrix} B & N \end{bmatrix},$$

and that c and x can be partitioned similarly as

$$c = \begin{pmatrix} c_{\mathcal{B}} \\ c_{\mathcal{N}} \end{pmatrix}, \qquad x = \begin{pmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{pmatrix}.$$

Show that the dictionary associated with the basis $\mathcal{B}$ can be written as

$$\begin{aligned} \eta &= \eta^* - z_{\mathcal{N}}^{*T} x_{\mathcal{N}}, \\ x_{\mathcal{B}} &= x_{\mathcal{B}}^* - B^{-1} N x_{\mathcal{N}}, \end{aligned}$$

where

$$\eta^* = c_{\mathcal{B}}^T B^{-1} b, \quad z_{\mathcal{N}}^* = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}, \quad x_{\mathcal{B}}^* = B^{-1} b.$$

What is the (primal) basic solution and objective value associated with this dictionary?

Answer: From the constraints $Ax = b$ it follows that

$$Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b,$$

and this gives the result for $x_{\mathcal{B}}$ since

$$x_{\mathcal{B}} = B^{-1}\Big(b - Nx_{\mathcal{N}}\Big) = B^{-1}b - B^{-1}Nx_{\mathcal{N}}.$$

Regarding the objective function,

$$\begin{aligned} \eta = c^T x &= c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T \Big(B^{-1}b - B^{-1}Nx_{\mathcal{N}}\Big) + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T B^{-1} b - c_{\mathcal{B}}^T B^{-1} N x_{\mathcal{N}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T B^{-1} b - \Big((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\Big)^T x_{\mathcal{N}}. \end{aligned}$$

The basic solution is obtained by setting $x_{\mathcal{N}} = 0$, which gives

$$x_{\mathcal{N}}^* = 0, \qquad x_{\mathcal{B}}^* = x_{\mathcal{B}}^*, \qquad \eta^* = \eta^*.$$

3c

Consider the LP problem

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 + 3x_3, \\ \text{subject to} \quad & \\ & 2x_1 + 3x_2 + x_3 \le 5, \\ & 4x_1 + x_2 + 2x_3 \le 11, \\ & 3x_1 + 4x_2 + 2x_3 \le 8, \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

(Continued on page 8.)

It turns out the optimal dictionary for this problem is

$$\begin{aligned}\max \eta &= 13 - 3x_2 - x_4 - x_6,\\ x_3 &= 1 + x_2 + 3x_4 - 2x_6,\\ x_1 &= 2 - 2x_2 - 2x_4 + x_6,\\ x_5 &= 1 + 5x_2 + 2x_4.\end{aligned}$$

For this dictionary, identify $\mathcal{B}$, $\mathcal{N}$, η^*, $x^*_{\mathcal{B}}$, $B^{-1}N$, $z^*_{\mathcal{N}}$.

Suppose the coefficient of 4 on x_2 in the objective function is changed to $4+t$ for some number $t > 0$. How large can t be chosen without sacrificing the optimality of the dictionary?

<u>Answer:</u> We read off the dictionary:

$$\begin{aligned}&\mathcal{B} = \{3,1,5\}, \qquad \mathcal{N} = \{2,4,6\},\\ &\eta^* = 13, \qquad x^*_{\mathcal{B}} = \begin{pmatrix}1\\2\\1\end{pmatrix},\\ &B^{-1}N = \begin{pmatrix}-1 & -3 & 2\\ 2 & 2 & -1\\ -5 & -2 & 0\end{pmatrix}, \qquad z^*_{\mathcal{N}} = \begin{pmatrix}3\\1\\1\end{pmatrix}.\end{aligned}$$

Next, changing 4 to $4+t$ changes $z^*_{\mathcal{N}}$ to

$$\bar{z}^*_{\mathcal{N}} := z^*_{\mathcal{N}} - t\begin{pmatrix}1\\0\\0\end{pmatrix},$$

while the other quantities remain unchanged. To ensure the optimality of dictionary t must be chosen such that $\bar{z}^*_{\mathcal{N}} \geq 0 \iff 3 - t \geq 0$, i.e.,

$$t \leq 3.$$

THE END