UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3100 — Linear Optimization

Day of examination: Tuesday, June 12th, 2018

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Consider the LP problem

$$
\begin{aligned}
&\max\ 5x_1 + 10x_2 \\
&\text{subject to} \\
&x_1 + 3x_2 \leq 50, \\
&4x_1 + 2x_2 \leq 60, \\
&x_1 \leq 5, \\
&x_1, x_2 \geq 0.
\end{aligned} \tag{1}
$$

Determine the dual problem linked to (1). Write (1) and the dual problem in matrix form (with inequality constraints).

b

State the complementary slackness theorem for a general LP problem. Suppose $(x_1, x_2) = (5, 15)$ is an optimal solution to (1). Use the complementary slackness theorem to solve the dual problem of (1).

c

(i) What is the definition of a convex set $C \subset \mathbb{R}^n$.
(ii) Let $f : C \to \mathbb{R}$ be a convex continuous function. What does it mean (definition) that f is convex? Illustrate your definition with a figure.

d

(i) Prove that a non-empty set $S \subset \mathbb{R}^n$ of optimal solutions to a LP problem is convex.

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(ii) Consider a LP problem with two optimal solutions x^1 and x^2. Explain that this problem must in fact possess infinitely many optimal solutions.

Problem 2

a

Determine the pure minmax and maxmin strategies for the game given by

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -3 & 2 \\ 1 & -2 & -2 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

Does the game have a value? You must justify your answers (include definitions of the involved concepts).

b

Consider the game called *Odd-or-Even*. The row and column players simultaneously call out one of the numbers 1 or 2. The row player wins if the sum of the numbers is odd. The column player wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in kroner.

This is an example of a two-person zero-sum game, so that the payoff of the column player is the negative of the payoff of the row player. We will therefore restrict attention to the payoff matrix of the row player, which is denoted by A. For the Odd-or-Even game the payoff matrix becomes

$$A = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}, \tag{2}$$

where a negative amount means that the row player pays the absolute value of this amount to the column player.

What do we mean by a saddle point of a general game $A = \{a_{i,j}\} \in \mathbb{R}^{m\times n}$? Does the Odd-or-Even game (2) possess a saddle point?

c

We consider mixed (randomized) strategies $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of the Odd-or-Even game (2), given by two numbers $p, q \in (0, 1)$. The row player chooses $i = 1$ with probability $x_1 = p \in (0, 1)$. The column player chooses $j = 1$ with probability $y_1 = q \in (0, 1)$.

Solve the Odd-or-Even game by finding "equalizing strategies", that is, determine p such that if the row player chooses $i = 1$ with probability p, then the average payoff of the row player is the same whether the column player chooses $j = 1$ or $j = 2$. Compute the average payoff of the row player (with the probability p that you found).

Formulate the analogous principle for the column player, and use it to determine the probability q. Compute the average payoff of the column player (with the probability q that you found).

Is the game is fair?

(Continued on page 3.)

Problem 3

Consider the LP problem

$$\begin{aligned} &\max\ x_1 + 2x_3 \\ &\text{subject to} \\ &x_1 + 2x_2 + x_3 \le 2, \\ &x_3 \le 1, \\ &x_1, x_2, x_3 \ge 0. \end{aligned} \tag{3}$$

a

Use the simplex method to solve (3).

b

Prove that if x^* and y^* are feasible for the primal and dual problems, respectively, and the corresponding objective values coincide, then x^* and y^* are optimal for their respective problems.

c

Identify the dual problem linked to the LP problem

$$\begin{aligned} &\max 4x_1 + 5x_2 + 6x_3, \\ &x_1 + x_3 \le 1, \\ &x_1 + x_2 \le 2, \\ &x_2 + x_3 \le 3, \\ &x_1, x_2, x_3 \ge 0. \end{aligned} \tag{4}$$

Without explicitly solving the problems, show that $(x_1, x_2, x_3) = (0, 2, 1)$ is the optimal solution of (4) and that $(y_1, y_2, y_3) = (\frac{5}{2}, \frac{3}{2}, \frac{7}{2})$ is the optimal solution of the dual problem.

THE END