UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3100 — Linear optimization

Day of examination: 0900, 3 June 2020 – 0900, 10 June 2020

This problem set consists of 3 pages.

Appendices: None

Permitted aids: All

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 8 part questions will be weighted equally.

Problem 1 Simplex method

Consider the LP problem

$$\begin{array}{llll} \text{maximize} & -x_1 & +3x_2 & \\ \text{subject to} & -x_1 & +x_2 & \leq 1, \\ & x_1 & & \leq 4, \\ & & x_2 & \leq 3, \\ & x_1, x_2 \geq 0. & & \end{array}$$

1a

Solve this using the simplex method with initial feasible solution $(x_1, x_2) = (0, 0)$. Find an optimal solution and corresponding optimal objective value.

1b

Illustrate the problem geometrically. Draw the feasible region and the contour line of the objective function $f(x_1, x_2) = -x_1 + 3x_2$ that passes through the optimal solution.

Problem 2 Standard form

Convert the LP problem

(Continued on page 2.)

$$\begin{array}{lrrrl} \text{maximize} & 3x_1 & +2x_2 & +4x_3 & \\ \text{subject to} & 2x_1 & -5x_2 & & = 6, \\ & -x_1 & & +3x_3 & \geq 4, \\ & & x_1, x_2 \geq 0 & & \end{array}$$

into standard form (the form suitable for the simplex algorithm). Note that $x_3 \in \mathbb{R}$ is a free variable. What form of the simplex algorithm will be required to solve it? (do not try to solve it).

Problem 3 Duality

Consider the LP problem

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j, \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m, \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n, \end{aligned}$$

and its dual

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i, \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, \ldots, n, \\ & y_i \geq 0, \quad i = 1, 2, \ldots, m. \end{aligned}$$

3a

Let w_i be the i-th slack variable in the primal problem, $i = 1, 2, \ldots, m$, and let z_j be the j-th slack variable in the dual problem, $j = 1, 2, \ldots, n$. Derive the following identity:

$$\sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} w_i y_i + \sum_{j=1}^{n} z_j x_j, \tag{1}$$

and use it to prove the Weak Duality Theorem.

3b

Recall that the Strong Duality Theorem states that if (P) has an optimal solution x^* then (D) has an optimal solution y^* and that

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

State the Complementary Slackness Theorem and and prove it *using the identity* (1).

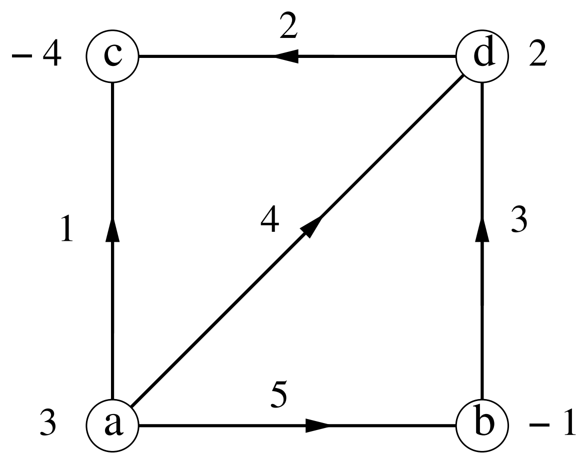
(Continued on page 3.)

3c

What is the optimal solution to the dual of the LP problem in Problem 1?

Problem 4 Network flow

Consider the minimum cost network flow problem based on the directed graph shown in the figure. The number associated with each directed edge



(i, j) is its cost $c_{i,j}$ (per unit flow). The number associated with each node i is the supply b_i.

4a

Let T_1 be the spanning tree consisting of the edges

$$(a, b), \quad (a, d), \quad (d, c).$$

Compute the tree solution x corresponding to T_1.

4b

Use the network simplex method to find an optimal solution and optimal value for the flow problem.

Good luck!