Oblig 1

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Exercise 1

Consider the problem

$$
\begin{array}{rcrcr}
\max\ 2x_1 & + & 3x_2 & & \\
\text{s.t. } 2x_1 & + & x_2 & \leq & 10 \\
x_1 & + & 2x_2 & \leq & 10 \\
x_1 & + & x_2 & \leq & 6 \\
 & & x_1, x_2 & \geq & 0
\end{array}
$$

a)

Write down a matrix A and vectors $\mathbf{b}$ and $\mathbf{c}$ so that the problem can be written on the form

$$
\begin{aligned}
\max\ & \mathbf{c}^T\mathbf{x} \\
\text{s.t. } & A\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

Solution: We get that

$$
A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 10 \\ 10 \\ 6 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}
$$

Some of you mixed this up with the representation of the simplex method stated in chapter 6, where the identity matrix is padded horizontally to the matrix (giving a 3×5-matrix), and slack variables are taken in. But $\mathbf{x}$ was here a vector with just two elements (x_1 and x_2), so that A can have only two columns.

b)

Sketch the feasible region of the problem.
Solution: The feasible region can be drawn by connecting the points $(0,0)$, $(5,0)$, $(4,2)$, $(2,4)$, $(0,5)$, and $(0,0)$ sequentially by line segments, and highlighting its interior. Many of you made an imprecise plot in the sense that the line segment from $(4,2)$ to $(2,4)$ was not shown. The points $(0,0)$, $(5,0)$, $(4,2)$, $(2,4)$, $(0,5)$ are the extreme points of the feasible region. They can also be found as the basic feasible solutions to the problem by going through all possible choices of bases (I did not require this here though), of which some are feasible and others not.

c)

Solve the problem using the Simplex method.
Solution: The initial dictionary is

$$\begin{array}{rcrcrcr} \eta & = & 0 & + & 2x_1 & + & 3x_2 \\ \hline w_1 & = & 10 & - & 2x_1 & - & x_2 \\ w_2 & = & 10 & - & x_1 & - & 2x_2 \\ w_3 & = & 6 & - & x_1 & - & x_2 \end{array}$$

If we apply the largest coefficient rule, x_2 will be the entering variable. The ratios are 1/10, 1/5, and 1/6. The biggest is 1/5, so that w_2 is the leaving variable. Rewriting the second constraint as $x_2 = 5 - \frac{1}{2}x_1 - \frac{1}{2}w_2$, and inserting this in the dictionary, we obtain

$$\begin{array}{rcrcrcr} \eta & = & 15 & + & 0.5x_1 & - & 1.5w_2 \\ \hline w_1 & = & 5 & - & 1.5x_1 & + & 0.5w_2 \\ x_2 & = & 5 & - & 0.5x_1 & - & 0.5w_2 \\ w_3 & = & 1 & - & 0.5x_1 & + & 0.5w_2 \end{array}$$

Now x_1 is entering, and the rations are 0.3, 0.1, and 0.5. 0.5 is biggest so that w_3 is leaving. We rewrite the third constraint as $x_1 = 2 - 2w_3 + w_2$, and insert in the dictionary:

$$\begin{array}{rcrcrcr} \eta & = & 16 & - & w_3 & - & w_2 \\ \hline w_1 & = & 2 & + & 3w_3 & - & w_2 \\ x_2 & = & 4 & + & w_3 & - & w_2 \\ x_1 & = & 2 & - & 2w_3 & + & w_2 \end{array}$$

This is optimal, with value 16. The basic solution is $w_2 = w_3 = 0$, and $w_1 = 2$, $x_2 = 4$, $x_1 = 2$, so that $\mathbf{x} = (2, 4)$.

d)

We consider the same constraints as in a), but change the objective function to $2x_1 + 2x_2$. Apply the simplex method again to find all optimal solutions to this modified problem.
Solution: The initial dictionary is now

$$\begin{array}{rcrcrcr} \eta & = & 0 & + & 2x_1 & + & 2x_2 \\ \hline w_1 & = & 10 & - & 2x_1 & - & x_2 \\ w_2 & = & 10 & - & x_1 & - & 2x_2 \\ w_3 & = & 6 & - & x_1 & - & x_2 \end{array}$$

It is tempting to choose the same entering variable as in c), to save calculations (the point being that the new coefficient is still positive). Thus, again we insert $x_2 = 5 - \frac{1}{2}x_1 - \frac{1}{2}w_2$ in the dictionary. This leads to the same modifications for the constraints as before. Only the objective changes from what we did above:

$$\begin{array}{rcrcrcr} \eta & = & 10 & + & x_1 & - & w_2 \\ \hline w_1 & = & 5 & - & 1.5x_1 & + & 0.5w_2 \\ x_2 & = & 5 & - & 0.5x_1 & - & 0.5w_2 \\ w_3 & = & 1 & - & 0.5x_1 & + & 0.5w_2 \end{array}$$

x_1 is also now entering, and we thus get the same leaving variable w_3. The rewriting $x_1 = 2 - 2w_3 + w_2$ thus again only changes the objective:

$$\begin{array}{rcrcrcr} \eta & = & 12 & - & 2w_3 & & \\ \hline w_1 & = & 2 & + & 3w_3 & - & w_2 \\ x_2 & = & 4 & + & w_3 & - & w_2 \\ x_1 & = & 2 & - & 2w_3 & + & w_2 \end{array}$$

This is optimal, with value 12. The basic solution is $w_2 = w_3 = 0$, and $w_1 = 2$, $x_2 = 4$, $x_1 = 2$, so that $\mathbf{x} = (2, 4)$ is a solution also now. This is not unique, however, since we can increase w_2 without changing the objective value. Let us see how much we can increase w_2 while maintaining primal feasibility:

- In the first constraint w_2 can be increased to 2.
- In the second constraint w_2 can be increased to 4.
- In the third constraint w_2 can be increased to infinity.

We thus see that w_2 can be increased to 2. Any point $(x_1, x_2) = (2+w_2, 4-w_2)$, with $0 \leq w_2 \leq 2$, is thus an optimal solution. Many of you found the optimal solutions $(2, 4)$ and $(4, 2)$, but forgot to mention that all points on the line segment between these two also are optimal.

Exercise 2

Find any optimal solution to the problem

$$\begin{array}{rcrcrcl} \max x_1 & + & x_2 & + & x_3 & & \\ \text{s.t. } 2x_1 & - & 2x_2 & + & x_3 & \leq & 4 \\ 3x_1 & - & x_2 & + & 2x_3 & \leq & 2 \\ & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Solution: The initial dictionary is

$$\begin{array}{rcrcrcrcr} \eta & = & & & x_1 & + & x_2 & + & x_3 \\ \hline w_1 & = & 4 & - & 2x_1 & + & 2x_2 & - & x_3 \\ w_2 & = & 2 & - & 3x_1 & + & x_2 & - & 2x_3 \end{array}$$

Any x_i can be chosen as entering variable, but we see that the ratios for x_2 are all negative (if we choose that one), and this implies that the problem is unbounded, so that there is no optimal solution.

Exercise 3

In the field of compressive sensing one attempts to recover an unknown vector from an underdetermined set of (linear) measurements, i.e., find an unknown $\mathbf{x} \in \mathbb{R}^N$ that satisfies $A\mathbf{x} = \mathbf{p}$, where

- $\mathbf{p} \in \mathbb{R}^m$ is the vector of measurements, and
- A is the $m \times N$ matrix which collects those measurements.

In practical applications m is much smaller than N, and we can't expect to recover $\mathbf{x}$ in general. But if we have some additional information about $\mathbf{x}$, it turns out that the knowledge of the measurements in $\mathbf{p}$ may still be enough to recover $\mathbf{x}$. The additional information we will consider is *sparsity* (a vector is called sparse if it has mostly components that are zero): For many "magic" matrices A, if it is known that $\mathbf{x}$ is sparse, $\mathbf{x}$ can be recovered as the optimal solution to the problem

$$\begin{array}{ll} \min & \|\mathbf{x}\|_1 \\ \text{s.t.} & A\mathbf{x} = \mathbf{p}, \end{array} \tag{1}$$

where $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \ldots + |x_N|$. Note that the variable $\mathbf{x}$ here is unconstrained: It is not required to be non-negative. In the following we will test if this procedure works for a very small vector and matrix.

a)

Show that $\mathbf{x}$ is an optimal solution to (1) if and only if it is an optimal solution to

$$\begin{array}{ll} \max & \sum_i(-x_i^+ - x_i^-) \\ \text{s.t.} & \begin{pmatrix} A & -A \\ -A & A \end{pmatrix} \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{pmatrix} \leq \begin{pmatrix} \mathbf{p} \\ -\mathbf{p} \end{pmatrix} \\ & \mathbf{x}^+, \mathbf{x}^- \geq \mathbf{0} \end{array} \tag{2}$$

This is a linear programming problem in standard form.
Hint: Write $x_i = x_i^+ - x_i^-$, where $x_i^+, x_i^- \geq 0$.
Solution: First rewrite the problem as

$$\begin{array}{ll} \min & \sum_i(x_i^+ + x_i^-) \\ \text{s.t.} & A(\mathbf{x}^+ - \mathbf{x}^-) = \mathbf{p} \\ & \mathbf{x}^+, \mathbf{x}^- \geq \mathbf{0} \end{array}$$

The decomposition $x_i = x_i^+ - x_i^-$ with $x_i^+, x_i^- \geq 0$ is not unique (something most of you did not address), but the minimum above is obtained when at least one of x_i^+ and x_i^- is zero, and then $x_i^+ + x_i^- = |x_i|$, so that the minimum obtained over all such decompositions is indeed a 1-norm.

We now write $A(\mathbf{x}^+ - \mathbf{x}^-) = \mathbf{p}$ as

$$\begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{pmatrix} = \mathbf{p},$$

and write this in terms of two inequalities

$$\begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{pmatrix} \leq \mathbf{p} \qquad \begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{pmatrix} \geq \mathbf{p}.$$

We multiply the second inequlity with -1 to turn it into a $\leq$-inequality, and stack the two sets of inequalities together to obtain

$$\begin{pmatrix} A & -A \\ -A & A \end{pmatrix} \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{pmatrix} \leq \begin{pmatrix} \mathbf{p} \\ -\mathbf{p} \end{pmatrix}$$

Let us test this procedure on the sparse vector $\mathbf{x} = (0, 0, -1)$, and the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$, to see if $\mathbf{x}$ is recovered from the vector of measurements, which here can be computed to be $\mathbf{p} = (1, 1)$. Of course, it does not sound like rocket science to recover a vector with 3 components from two measurements. But the magic is that solving the problem (1) also can help recover sparse vectors $\mathbf{x}$ when N is very large and m is very small compared to N!

b)

Solve (2), with A and $\mathbf{p}$ as given above, using the simplex method. Is the correct $\mathbf{x}$ recovered? Is the optimum unique?

To get started, you can use that the primal dictionary is

ζ	$=$		$-x_1$	$-x_2$	$-x_3$	$-x_4$	$-x_5$	$-x_6$
w_1	$=$	1	$-x_1$		$+x_3$	$+x_4$		$-x_6$
w_2	$=$	1		$-x_2$	$+x_3$		$+x_5$	$-x_6$
w_3	$=$	-1	$+x_1$		$-x_3$	$-x_4$		$+x_6$
w_4	$=$	-1		$+x_2$	$-x_3$		$-x_5$	$+x_6$

where we wrote $\mathbf{x}^+ = (x_1, x_2, x_3)$, $\mathbf{x}^- = (x_4, x_5, x_6)$, and denoted the slack variables by w_i.

Hint: The starting dictionary above is not primal feasible, but dual feasible. So write down the dual dictionary or apply the dual simplex method.

Solution: Taking the negative transpose the dual dictionary is

$-\eta$	$=$		$-y_1$	$-y_2$	$+y_3$	$+y_4$
z_1	$=$	1	$+y_1$		$-y_3$	
z_2	$=$	1		$+y_2$		$-y_4$
z_3	$=$	1	$-y_1$	$-y_2$	$+y_3$	$+y_4$
z_4	$=$	1	$-y_1$		$+y_3$	
z_5	$=$	1		$-y_2$		$+y_4$
z_6	$=$	1	$+y_1$	$+y_2$	$-y_3$	$-y_4$

Choosing y_3 as the entering variable and z_1 as leaving yields $y_3 = 1 + y_1 - z_1$. This gives the new dictionary

$-\eta$	$=$	1		$-y_2$	$-z_1$	$+y_4$
y_3	$=$	1	$+y_1$		$-z_1$	
z_2	$=$	1		$+y_2$		$-y_4$
z_3	$=$	2		$-y_2$	$-z_1$	$+y_4$
z_4	$=$	2			$-z_1$	
z_5	$=$	1		$-y_2$		$+y_4$
z_6	$=$			y_2	$+z_1$	$-y_4$

This dictionary is degenerate due to the last constaint. Choosing y_4 as entering variable and z_6 as leaving (degenerate pivot) yields $y_4 = y_2 + z_1 - z_6$. This gives the new dictionary

$-\eta$	$=$	1				$-z_6$
y_3	$=$	1	$+y_1$		$-z_1$	
z_2	$=$	1			$-z_1$	$+z_6$
z_3	$=$	2				$-z_6$
z_4	$=$	2			$-z_1$	
z_5	$=$	1			$+z_1$	$-z_6$
y_4	$=$			y_2	$+z_1$	$-z_6$

which is optimal. The corresponding primal dictionary is

$$
\begin{array}{rcrrrrrrr}
\zeta & = & -1 & -w_3 & -x_2 & -2x_3 & -2x_4 & -x_5 & \\
\hline
w_1 & = & & -w_3 & & & & & \\
w_2 & = & & & & & & & -w_4 \\
x_1 & = & & w_3 & +x_2 & & +x_4 & -x_5 & -w_4 \\
x_6 & = & 1 & & -x_2 & +x_3 & & +x_5 & +w_4
\end{array}
$$

From this it is clear that $x_2 = x_3 = x_4 = x_5 = w_3 = 0$. From the second constraint it also follows that the last nonbasic variable w_4 must be zero. The optimal value is -1. It follows that $\mathbf{x}^+ = (x_1, x_2, x_3) = (0,0,0)$, $\mathbf{x}^- = (x_4, x_5, x_6) = (0,0,1)$ is the unique optimum, so that $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^- = (0,0,-1)$. Thus, the correct $\mathbf{x}$ was recovered. Many of you were able to solve the simplex method, but did not map the result back to $\mathbf{x}^+$, $\mathbf{x}^-$, and finally $\mathbf{x}$.

A couple of remarks should be made.

- For this particular exercise, simplex is not the easiest way to solve (1): That $A\mathbf{x} = \mathbf{p}$ means simply that $x_1 - x_3 = x_2 - x_3 = 1$, so that $x_1 = x_2 = x_3 + 1$, with x_3 arbitrary. The problem thus boils down to minimizing $|x_1| + |x_2| + |x_3| = 2|x_3| + |x_3 + 1|$, which is easily solved by hand.
- For larger A and $\mathbf{p}$, we depend on an implementation of simplex, since such problems are too tedious to solve by hand.