Welcome to MAT 3100

- pdf's from lectures will be placed in the cours schedale.
- All lectures in english
- Lectures: $14^{15}-16^{00}$ mondays : Exercises (excpet first week(s))
  $14^{15}-16^{00}$ tuesdays.
- Exercises: Attempt by hand first.
- Compulsory exercises: Deadlines 13/3, 24/4
- Cours webpage: Code and other resources. Curriculum.

Lecture 1: Linear optimization is about maximizing a linear function in several variables, subject to constraints that are linear (in)equalities.

Example 1: Resource allication problem.

Notation: $i = 1, \dots, m$: raw materials
$j = 1, \dots, n$: products

a_{ij} : Required amount of raw material i to produce one unit of product j.

σ_j : price per unit of product j (company sells at this price)

b_i : units available of raw material i

ρ_i : price per unit of raw material i (on market)

x_j : units of product j to produce (≥ 0, unknowns)

$$c_j = \underbrace{\sigma_j}_{\text{price/unit}} - \underbrace{\sum_{i=1}^{m} \rho_i a_{ij}}_{\substack{\text{price raw materials} \\ \text{to produce one unit of} \\ \text{product } j}} = \text{profit for producing one unit of product } j.$$

$\sum_{j=1}^{n} c_j x_j = \vec{c}^T \vec{x}$ Total profit.

$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ constraints (together with $x_j \geq 0$)

a_{ij}: raw material i needed for all products

b_i: availability

Question: How many units of each product should we produce in order to maximize total profit?

So: Which $\vec{x}$ maximizes $\sum_{j=1}^{n} c_j x_j$ s.a. $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $\vec{x} \geq 0$.

Matrix form:

$$\begin{aligned} \max \quad & \vec{c}^T \vec{x} \\ \text{s.t.} \quad & A\vec{x} \leq \vec{b} \\ & \vec{x} \geq \vec{0} \end{aligned}$$

Standard form for linear programming (LP) problem.

$\geq$, $\leq$: componentwise.

primal problem

$$\updownarrow$$

$$\begin{aligned} \min \quad & \vec{b}^T \vec{y} \\ \text{s.t.} \quad & A^T \vec{y} \geq \vec{c} \\ & \vec{y} \geq \vec{0} \end{aligned}$$

dual problem

Example 2 Let w_i be market price (we sell for) raw material i (unknown)

$\sum_{i=1}^{m} b_i w_i$: "lost opportunity cost" of sitting on the raw material (want to be small).

We want to minimize $\sum_{i=1}^{m} b_i w_i$

Constraints: $w_i \geq p_i$ (else everyone would instead buy raw materials from us).

$\sum_{i=1}^{m} w_i a_{ij} \geq \sigma_j$ ($\sum_{i=1}^{m} w_i a_{ij}$: price for selling raw materials needed; else other companies can buy raw materials from us, produce product j, and sell it cheaper than us.

to produce one unit of product j

Set $y_i = w_i - p_i \geq 0$ (so that $w_i = y_i + p_i$)

Minimize lost opportunity cost:

$$\min \sum_{i=1}^{m} b_i w_i = \sum_{i=1}^{m} b_i (y_i + p_i) = \sum_{i=1}^{m} b_i y_i + \sum_{i=1}^{m} b_i p_i$$

($\sum_{i=1}^{m} b_i y_i$: unknown; $\sum_{i=1}^{m} b_i p_i$: constant)

constraints: $\sum_{i=1}^{m} w_i a_{ij} \geq \sigma_j$

$$\Downarrow$$

$$\sum_{i=1}^{m} y_i a_{ij} \geq \sigma_j - \sum_{i=1}^{m} p_i a_{ij} = c_j$$

In summary:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i a_{ij} \geq c_j \\ & \vec{y} \geq \vec{0} \end{aligned}$$

This is the component form of the dual problem, so that the problems in Example 1 and 2 are primal and dual problems

Terminology In

$$\begin{aligned} \max \quad & \vec{c}^{\,T} \vec{x} \\ \text{s.t.} \quad & A\vec{x} \leq \vec{b} \\ & \vec{x} \geq \vec{0} \end{aligned}$$

1. $f(\vec{x}) = \vec{c}^{\,T} \vec{x}$ is called the objective function.
2. An $\vec{x}$ which satisfies the constraints $A\vec{x} \leq \vec{b}$, $\vec{x} \geq \vec{0}$, is called a feasible solution.
3. The problem is called unbounded if there exist $\vec{x}$ with arbitrarily high objective value.

Questions: 1. Is there a feasible $\vec{x}$?

2. Is the problem unbounded

3. Is there a unique maximizer? How do we find it.

Such an $\vec{x}$ is called an optimal solution

Note: Inequality constraints can be converted to equality constraints by introducing slack variable $w_1, w_2, \ldots$

$$A\vec{x} \le \vec{b} \iff \vec{w} := \vec{b} - A\vec{x} \ge \vec{0}$$

$$A\vec{x} + \vec{w} = \vec{b} \iff [A\ I]\begin{bmatrix}\vec{x}\\ \vec{w}\end{bmatrix} \le \vec{b}$$

Example from chapter 2

Inequality form

$$\begin{aligned} \max.\quad & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.}\quad & 2x_1 + 3x_2 + x_3 \le 5 \quad (w_1) \\ & 4x_1 + x_2 + 2x_3 \le 11 \quad (w_2) \\ & 3x_1 + 4x_2 + 2x_3 \le 8 \quad (w_3) \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Equality form (dictionary form)

$$\begin{aligned} \max\quad & \xi = 0 + 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.}\quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \ge 0 \end{aligned}$$

Note: As long as $b_i \ge 0$ (here 5, 11, 8), we can get an initial feasible solution by setting $x_1 = x_2 = x_3 = 0$ ($\Rightarrow w_1 = 5, w_2 = 11, w_3 = 8$)

Note: If one c_j in the objective is > 0, then the objective can be increased by increasing x_j

Rule of thumb: Increase the x_j with largest c_j, here x_1.
Keep the others at 0.
w_1, w_2, w_3 will then decrease, until one of them becomes 0.

First eq. $5 - 2x_1 = 0 \Rightarrow x_1 = \frac{5}{2} = 2.5$ (smallest)

Second eq. $11 - 4x_1 = 0 \Rightarrow x_1 = \frac{11}{4} = 2.75$

third eq. $8 - 3x_1 = 0 \Rightarrow x_1 = \frac{8}{3} \approx 2.66$

We can only increase x_1 to 2.5, to preserve feasibity.

$x_1 = 2.5, x_2 = x_3 = 0 \Rightarrow w_1 = 0$

Let x_1 and w_1 swap roles: $w_1 = 5 - 2x_1 - 3x_2 - x_3$

$x_1 = \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3$

insert everywhere.

$\xi = 0 + 5x_1 + 4x_2 + 3x_3 = 0 + 5(\frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3) + 4x_2 + 3x_3 = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$

$x_1 = \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3$

$w_2 = 11 - 4x_1 - x_2 - 2x_3 = 11 - 4(\frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3) - x_2 - 2x_3 = 1 + 2w_1 + 5x_2$

$w_3 = 8 - 3x_1 - 4x_2 - 2x_3 = 8 - 3(\frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3) - 4x_2 - 2x_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3$

New feasible solution: $w_1 = x_2 = x_3 = 0 \Rightarrow x_1 = \frac{5}{2}, w_2 = 1, w_3 = \frac{1}{2}$

See now: Only an increase in x_3 can increase the objective.
keep w_1 and x_2 at 0, increase x_3

First eq: $\frac{5}{2} - \frac{1}{2}x_3 = 0 \Rightarrow x_3 = 5$

Second eq: 1

Third eq: $\frac{1}{2} - \frac{1}{2}x_3 = 0 \Rightarrow x_3 = 1$, (smallest)

We can thus increase x_3 to 1, then we get that $w_3 = 0$.

x_3 and w_3 now swap roles:

$$w_3 = \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3$$

$$\Updownarrow$$

$$x_3 = 1 + 3w_1 + x_2 - 2w_3$$

insert everywhere:

$$\xi = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3 = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}(1 + 3w_1 + x_2 - 2w_3) = 13 - w_1 - 3x_2 - w_3$$

$x_1 = \ldots\ldots$ $= 2 - 2w_1 - 2x_2 + w_3$

$w_2 =$ $= 1 + 2w_1 + 5x_2$

$x_3 =$ $= 1 + 3w_1 + x_2 - 2w_3$

New objective value improved to 13.

This can't be improved further, as all $c_j < 0$

We get an optimal solution by setting $w_1 = x_2 = w_3 = 0$

constraints give $x_1 = 2$, $x_3 = 1$, $w_2 = 1$

$$\Downarrow$$

$$\vec{x} = (x_1, x_2, x_3) = (2, 0, 1),$$

which is the unique optimal solution.