Lecture 21. January

Terminology

Dictionary: An equation system obtained along the way

Feasible dictionary: A dictionary where all $b_i \geq 0$

Variable on the right hand side: Nonbasic variables

the left hand side: Basic variables.

pivoting: Going from one dictionary to the next.

The simplex method in general

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \dots, m$$

$$x_j \geq 0 \quad j = 1, \dots, n$$

slack variables ($w_i = b_i - \sum_{j=1}^{n} a_{ij} x_j$), are initially basic (i.e. appear on left side)

In the book: $w_1 \longrightarrow x_{n+1}$, $\dots$, $w_m \longrightarrow x_{n+m}$

B: Indices of basic variables (position of $\{1, \dots, n+m\}$)

N: Indices of nonbasic variables

After some pivots, the new dictionary looks like

$$\xi = \bar{\xi} + \sum_{j=1}^{n} \bar{c}_j x_j$$

$$x_i = \bar{b}_i - \sum_{j=1}^{n} \bar{a}_{ij} x_j$$

new objective value

bars indicate value that change.

in one pivot, the

$\bar{\xi}$, $\bar{c}_j$, $\bar{a}_{ij}$, $\bar{b}_i$, B, N

change

Entering variable : The variable that goes from nonbasic to basic

Leaving variable : ——————— // ——— basic to nonbasic.

Note: If x_k is entering, then $\bar{c}_k > 0$

If x_i is leaving, then equation i in the system says that

$$x_i = \bar{b}_i - \bar{a}_{ik} x_k = 0 \iff x_k = \frac{\bar{b}_i}{\bar{a}_{ik}}$$

so, x_k can only be increased to this.

Note: If $a_{ik} \leq 0$, we can increase x_k to ∞ without violating the constraint.

In summary: 1. x_k can be increased to $\min \frac{\bar{b}_i}{\bar{a}_{ik}}$

2. Leaving variable must be picked from

$$\{ i \in B : \bar{a}_{ik} > 0, \frac{\bar{b}_i}{\bar{a}_{ik}} \text{ minimal} \}$$

may not be unique.

Alternative expression: ($\bar{b}_i = 0$: this is in chap. 3)

rewrite : $\bar{b}_i - \bar{a}_{ik} x_k \geq 0$ to

$$\frac{1}{x_k} \geq \frac{\bar{a}_{ik}}{\bar{b}_i}$$

①

$$\frac{1}{x_k} \geq \max \frac{\bar{a}_{ik}}{\bar{b}_i}$$

may be ≤ 0, unbounded case.

$$x_k \le \frac{1}{\max \frac{a_{ik}}{b_i}}$$

Thus: pick leaving variable from $\{i \in B : \frac{a_{ik}}{b_i} \text{ is maximal}\}$

Negative right hand sides b_i:

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \le b_i$$
$$x_i \ge 0$$

original problem may not be feasible.

$$\max -x_0$$
$$\text{s.t} \sum_{j=1}^{n} a_{ij} x_j - x_0 \le b_i$$
$$x_0, x_1, \dots, x_n \ge 0$$

auxiliary problem always feasible, just choose x_0 big enough.

Note: Original problem is feasible
$\Leftrightarrow$
auxiliary problem feasible with $x_0 = 0$

To solve the original problem when we don't have an initial feasible solution:

1. Write down the dictionary for the auxiliary problem
2. perform a pivot with x_0 entering variable, the w_i with the smallest b_i leaving (most infeasible inequality).

Example: $w_1 = -1 + \dots + x_0$

$w_2 = -2 + \dots + x_0 \Rightarrow x_0 = 2 + \dots + w_2$

most infeasible.

$w_1 = -1 + \dots + (2 + \dots + w_2)$

$= 1 + \dots + w_2$

feasible

the point is that the new dictionary will be feasible after this pivot.

3. Solve the new dictionary

If objective value is <0, the original problem is infeasible.

4. Otherwise, the original problem is feasible, set $x_0 = 0$ in the optimal dictionary for the auxiliary problem. This gives a feasible dictionary for the original problem, which we can solve.

phase 1: Deciding feasibility of original problem (by solving auxiliary problem).

phase 2: Go from a feasible solution to an optimal solution of the original problem.

Unboundedness if all $\frac{\bar{a}_{ik}}{\bar{b}_i} \leq 0$, none of the basic variables can become $0 \Rightarrow$ so that entering variable x_k can be increased to ∞.
Since $c_k > 0$ and $x_k \to \infty$, we have that objective value $\to \infty$, so that the problem is unbounded.

Algorithms for pivots / simplex:

We replace A with $-A$, so that

$$w_i = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$\downarrow$$

$$w_i = b_i + \sum_{j=1}^{n} a_{ij} x_j$$

Dictionary

$$\xi = \bar{\xi} + \bar{c}_1 x_1 + \cdots + \bar{c}_n \bar{x}_n$$
$$x_{n+1} = w_1 = \bar{b}_1 + \bar{a}_{11} x_1 + \cdots + \bar{a}_{1n} \bar{x}_n$$
$$\vdots$$
$$x_{n+m} = w_n = \bar{b}_m + \bar{a}_{m1} x_1 + \cdots + \bar{a}_{mn} x_n$$

$$\begin{pmatrix} \bar{\xi} & \bar{c}_1 & \cdots & \bar{c}_n \\ \bar{b}_1 & \bar{a}_{11} & \cdots & \bar{a}_{1n} \\ \vdots & & & \vdots \\ \bar{b}_m & \bar{a}_{m1} & & \bar{a}_{mn} \end{pmatrix}$$

If x_r leaves and x_s enters:

$$x_r = \bar{b}_r + \sum_{j=1}^{n} \bar{a}_{rj} x_j$$

$$\updownarrow$$

$$x_s = \frac{1}{\bar{a}_{rs}}\left(-\bar{b}_r - \sum_{\substack{j=1 \\ j \neq s}}^{n} \bar{a}_{rj} x_j + x_r\right)$$

$$[\bar{b}_r \ \bar{a}_{r1} \cdots \bar{a}_{rn}]$$

$$\downarrow$$

$$\frac{1}{\bar{a}_{rs}}[-\bar{b}_r \ -\bar{a}_{r1} \cdots \ \cdots -a_{rn}]$$

↑ in one place there is a 1

Thus, row r in the dictionary is transformed according to

$$\vec{a}_r \longrightarrow -\frac{1}{\bar{a}_{rs}}[\ b_r \ a_{r1} \cdots \ -1 \ a_{s+1} \ \cdots \ ]$$