Lecture 27. January

Exercises from chap. 1 and 2.

Exercise 1.1

x_1 : amount of bands (in tons) to produce this week

x_2 : " " coils " "

profit : $f(x_1, x_2) = 25x_1 + 30x_2$

constraints:

$$x_1 \leq 6000$$

$$x_2 \leq 4000$$

$$\frac{x_1}{200} + \frac{x_2}{140} \leq 40 \quad \Rightarrow x_2 \leq -0.7x_1 + 5600$$

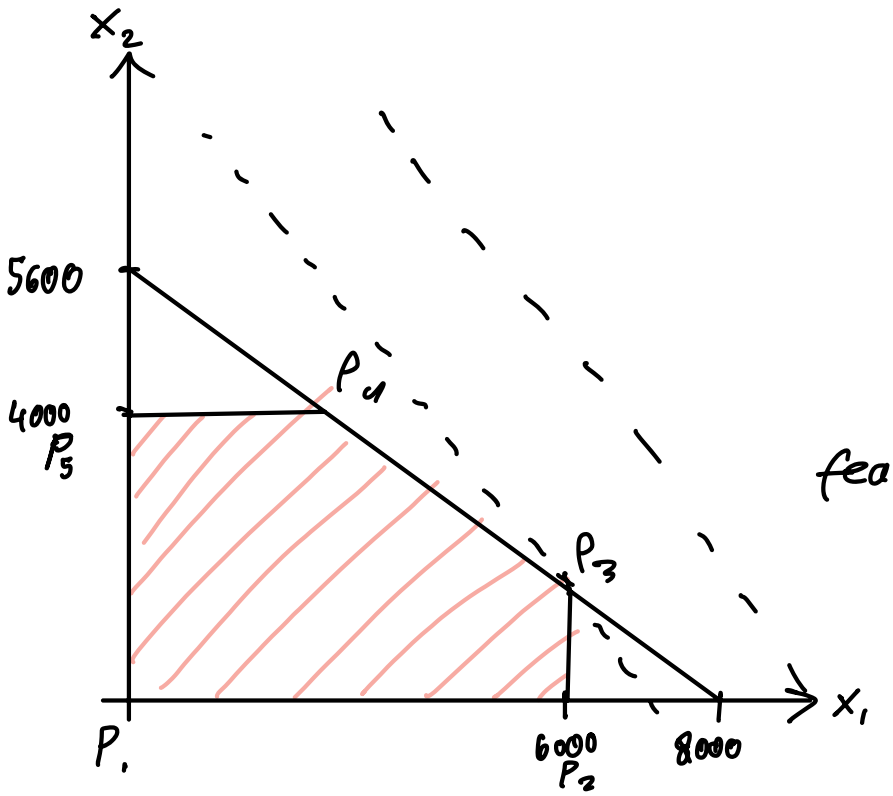
time to produce ($x_1, x_2 \geq 0$) x_1 bands, x_2 coils, in hours

equality for (0, 5600) (8000, 0)

contour curves of objective: $25x_1 + 30x_2 = \text{constant}$

$$x_2 = -\frac{5}{6}x_1 + \text{constant}$$

inclination: $-\frac{5}{6} \approx -0.8333$ steeper than contours of constraints



feasible region.

By inspection:

P_1 : (0, 0)

P_2 : (6000, 0)

$f(0,0) = 0$

$f(6000, 0) = 150\,000$

P_3 : intersection between
$x_1 = 6000$ and $\frac{x_1}{200} + \frac{x_2}{140} = 40$
$\Rightarrow x_2 = 1400$
$\Rightarrow (6000, 1400)$ $\quad f(6000, 1400) = 192000$

P_4 : intersection between
$x_2 = 4000$ and $\frac{x_1}{200} + \frac{x_2}{140} = 40$
$\Rightarrow x_1 = \frac{16000}{7}$ $\quad f(\frac{16000}{7}, 4000) \approx 177143$
$\Rightarrow (\frac{16000}{7}, 4000)$

P_5 : $(0, 4000)$ $\quad f(0, 4000) = 120000$

We see that we have maximum at P_3 : $f(6000, 1400) = 192000$

We solve this with simplex:

$$\xi = 25x_1 + 30x_2$$
$$w_1 = 6000 - x_1$$
$$w_2 = 4000 - x_2$$
$$w_3 = 40 - \frac{x_1}{200} - \frac{x_2}{140}$$

pivot (2, 2)

Leaving variable: 1.eq : 6000 (can't become 0)
2.eq : $4000 - x_2 = 0 \Rightarrow x_2 = 4000$, smallest
3.eq : $40 - \frac{x_2}{140} = 0 \Rightarrow x_2 = 5600$

w_2 is the leaving variable

$w_2 = 4000 - x_2 \Rightarrow x_2 = 4000 - w_2$

New system:

$$\xi = 25x_1 + 30(4000 - w_2) = 120000 + 25x_1 - 30w_2$$
$$w_1 = 6000 - x_1$$
$$x_2 = 4000 - w_2$$
$$w_3 = 40 - \frac{x_1}{200} - \frac{1}{140}(4000 - w_2) = \frac{80}{7} - \frac{x_1}{200} + \frac{w_2}{140}$$

x_1 enters

new basic solution: $x_1 = w_2 = 0$, $x_2 = 4000$

$P_5 (0, 4000)$.

Leaving var: 1. eq: $6000 - x_1 = 0 \Rightarrow x_1 = 6000$

2. eq: 4000 (can't become 0)

3. eq: $\frac{80}{7} - \frac{x_1}{200} = 0 \Rightarrow x_1 = \frac{16000}{7}$, smallest, w_3 leaves

pivot(1,3)

Exercise 2.2

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t} \quad & 2x_1 + x_2 \leq 4 \\ & 2x_1 + 3x_2 \leq 3 \\ & 4x_1 + x_2 \leq 5 \\ & x_1 + 5x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \zeta &= 2x_1 + x_2 \\ w_1 &= 4 - 2x_1 - x_2 \\ w_2 &= 3 - 2x_1 - 3x_2 \\ w_3 &= 5 - 4x_1 - x_2 \\ w_4 &= 1 - x_1 - 5x_2 \end{aligned}$$

x_1 enters.

Leaving: 1.eq: $4 - 2x_1 = 0 \Rightarrow x_1 = 2$

2.eq: $3 - 2x_1 = 0 \Rightarrow x_1 = \frac{3}{2}$

3.eq: $5 - 4x_1 = 0 \Rightarrow x_1 = \frac{5}{4}$

4.eq: $1 - x_1 = 0 \Rightarrow x_1 = 1$, smallest

(pivot(1,4)) w_4 leaves.

$x_1 = 1 - w_4 - 5x_2$

$$\begin{aligned} \zeta &= 2(1 - w_4 - 5x_2) + x_2 = 2 - 2w_4 - 9x_2 \\ w_1 &= 4 - 2(1 - w_4 + 5x_2) - x_2 = 2 + \dots \\ w_2 &= 3 - 2(1 - w_4 + 5x_2) - 3x_2 = 1 + \dots \\ w_3 &= 5 - 4(1 - w_4 + 5x_2) - x_2 = 1 + \dots \\ x_1 &= 1 - w_4 - 5x_2 \end{aligned}$$

← optimal dictionary can stop here.

$w_4 = x_2 = 0$, $x_1 = 1 \Rightarrow f(1,0) = 2$ is maximum

Exercise 2.1

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 + 3x_4 \le 5 \\ & x_1 + 3x_2 + x_3 + 2x_4 \le 3 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$$\begin{aligned} \xi &= 6x_1 + 8x_2 + 5x_3 + 9x_4 \quad x_4 \text{ enters} \\ w_1 &= 5 - 2x_1 - x_2 - x_3 - 3x_4 \\ w_2 &= 3 - x_1 - 3x_2 - x_3 - 2x_4 \end{aligned}$$

pivot (4, 2)

Leaving: 1.eq: $5 - 3x_4 = 0 \Rightarrow x_4 = \frac{5}{3}$

2.eq: $3 - 2x_4 = 0 \Rightarrow x_4 = \frac{3}{2}$ smallest, w_2 leaves

$$w_2 = 3 - x_1 - 3x_2 - x_3 - 2x_4 \iff x_4 = \frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2$$

$$\begin{aligned} \xi &= 6x_1 + 8x_2 + 5x_3 + 9\left(\tfrac{3}{2} - \tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{2}w_2\right) = \tfrac{27}{2} + \tfrac{3}{2}x_1 - \tfrac{11}{2}x_2 + \tfrac{1}{2}x_3 - \tfrac{9}{2}w_2 \\ w_1 &= \cdots\cdots \qquad = \tfrac{1}{2} - \tfrac{1}{2}x_1 + \tfrac{7}{2}x_2 + \tfrac{1}{2}x_3 + \tfrac{3}{2}w_2 \\ x_4 &= \qquad\qquad = \tfrac{3}{2} - \tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{2}w_2 \end{aligned}$$

Second iteration: x_1 enters. pivot (1, 1)

Leaving: 1.eq: $\frac{1}{2} - \frac{1}{2}x_1 = 0 \Rightarrow x_1 = 1$, smallest, w_1 leaves

2.eq: $\frac{3}{2} - \frac{1}{2}x_1 = 0 \Rightarrow x_1 = 3$

$$w_1 = \tfrac{1}{2} - \tfrac{1}{2}x_1 + \tfrac{7}{2}x_2 + \tfrac{1}{2}x_3 + \tfrac{3}{2}w_2 \iff x_1 = 1 - 2w_1 + 7x_2 + x_3 + 3w_2$$

$$\begin{aligned} \xi &= \tfrac{27}{2} + \tfrac{3}{2}(1 - 2w_1 + 7x_2 + x_3 + 3w_2) - \tfrac{11}{2}x_2 + \tfrac{1}{2}x_3 - \tfrac{9}{2}w_2 = 15 - 3w_1 + 5x_2 + 2x_3 \\ x_1 &= \qquad\qquad = 1 - 2w_1 + 7x_2 + x_3 + 3w_2 \\ x_4 &= \cdots \qquad\qquad = 1 + w_1 - 5x_2 - x_3 - 2w_2 \end{aligned}$$

Third iteration: x_2 enters. pivot (2, 2)

Leaving: 1.eq: $1 + 7x_2 = 0$, never happens

2.eq: $1 - 5x_2 = 0 \Rightarrow x_2 = \frac{1}{5}$, smallest, x_4 leaves.

$$x_4 = 1 + w_1 - 5x_2 - x_3 - 2w_2 \iff x_2 = \frac{1}{5} + \frac{1}{5}w_1 - \frac{1}{5}x_4 - \frac{1}{5}x_3 - \frac{2}{5}w_2$$

$\xi = 15 - 3w_1 + 5\left(\frac{1}{5} + \frac{1}{5}w_1 - \frac{1}{5}x_4 - \frac{1}{5}x_3 - \frac{2}{5}w_2\right) = 16 - 2w_1 - x_4 + x_3 - 2w_2$

$x_1 = \ldots\ldots = \frac{12}{5} - \frac{3}{5}w_1 - \frac{7}{5}x_4 - \frac{2}{5}x_3 + \frac{1}{5}w_2$

$x_2 = = \frac{1}{5} + \frac{1}{5}w_1 - \frac{1}{5}x_4 - \frac{1}{5}x_3 - \frac{2}{5}w_2$

Fourth iteration: x_3 enters. pivot(3,2)

Leaving: 1.eq: $\frac{12}{5} - \frac{2}{5}x_3 = 0 \Rightarrow x_3 = 6$

2.eq: $\frac{1}{5} - \frac{1}{5}x_3 = 0 \Rightarrow x_3 = 1$, smallest, x_2 leaves

$x_2 = \frac{1}{5} + \frac{1}{5}w_1 - \frac{1}{5}x_4 - \frac{1}{5}x_3 - \frac{2}{5}w_2 \Leftrightarrow x_3 = 1 + w_1 - x_4 - 5x_2 - 2w_2$

$\xi = 16 - 2w_1 - x_4 + (1 + w_1 - x_4 - 5x_2 - 2w_2) - 2w_2 = 17 - w_1 - 2x_4 - 5x_2 - 4w_2$

$x_1 = \ldots = 2 + \ldots$

$x_3 = \ldots = 1 + \ldots$

This dictionary is optimal, optimal value is 17,

$w_1 = x_4 = x_2 = w_2 = 0$, $x_1 = 2$, $x_3 = 1$

optimal $\vec{x} = (x_1, x_2, x_3, x_4) = (2, 0, 1, 0)$

The objective value in the pivots: $0 \to 13.5 \to 15 \to 16 \to 17$.