Exam 2008, 1a

$$
\begin{aligned}
\max \quad & -x_1 + 2x_2 \\
\text{s.t.} \quad & -x_1 + x_2 \le 3 \\
& x_1 \le 5 \\
& x_2 \le 5 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\xi &= -x_1 + 2x_2 \\
w_1 &= 3 + x_1 - x_2 \\
w_2 &= 5 - x_1 \\
w_3 &= 5 - x_2
\end{aligned}
$$

First iteration: x_2 enters

Leaving:
- 1.eq: $3 - x_2 = 0 \Rightarrow x_2 = 3$, smallest, w_1 leaves
- 2.eq: $5 - x_1 = 0$, never becomes zero
- 3.eq: $5 - x_2 = 0 \Rightarrow x_2 = 5$

$$w_1 = 3 + x_1 - x_2 \iff x_2 = 3 + x_1 - w_1$$

$$
\begin{aligned}
\xi &= -x_1 + 2(3 + x_1 - w_1) = 6 + x_1 - 2w_1 \\
x_2 &= 3 + x_1 - w_1 \\
w_2 &= 5 - x_1 \\
w_3 &= 5 - (3 + x_1 - w_1) = 2 - x_1 + w_1
\end{aligned}
$$

Second iteration: x_1 enters.

Leaving:
- 1.eq: $3 + x_1 = 0$, never happens
- 2.eq: $5 - x_1 = 0 \Rightarrow x_1 = 5$
- 3.eq: $2 - x_1 = 0 \Rightarrow x_1 = 2$, smallest, w_3 leaves

$$w_3 = 2 - x_1 + w_1 \iff x_1 = 2 - w_3 + w_1$$

$\xi = 6 + (2 - w_3 + w_1) - 2w_1 = 8 - w_3 - w_1$

$x_2 = 3 + (2 - w_3 + w_1) - w_1 = 5 - w_3$

$w_2 = 5 - (2 - w_3 + w_1) = 3 + w_3 - w_1$

$x_1 = \quad\quad\quad\quad\quad\quad = 2 - w_3 + w_1$

This is optimal.
8 is optimal value
for $w_1 = w_3 = 0$
$x_2 = 5$
$w_2 = 3$
$x_1 = 2$

$\Rightarrow \vec{x} = (x_1, x_2) = (2, 5)$

Exam 2011 1a

$$\begin{aligned} \max \quad & x_1 - 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_3 \le 12 \\ & 2x_1 + x_2 - x_3 \le 6 \\ & -x_1 + 3x_2 \le 9 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

$\xi = x_1 - 2x_2 + x_3$

$w_1 = 12 - x_1 - 2x_2 - x_3$

$w_2 = 6 - 2x_1 - x_2 + x_3$

$w_3 = 9 + x_1 - 3x_2$

find max of these.
x_1 enters
ratios $\left(-\frac{a_{ik}}{b_i}\right)$
$1/12$
$1/3$, biggest $(\frac{1}{3})$, so w_2 leaves.
$-1/9$

$w_i = b_i + a_{ik} x_k = 0$

$x_k = -\frac{b_i}{a_{ik}}$

we have taken min of these.

$w_2 = 6 - 2x_1 - x_2 + x_3 \iff x_1 = 3 - \frac{1}{2} w_2 - \frac{1}{2} x_2 + \frac{1}{2} x_3$

ratios:

$\xi = (3 - \frac{1}{2} w_2 - \frac{1}{2} x_2 + \frac{1}{2} x_3) - 2x_2 + x_3 = 3 - \frac{1}{2} w_2 - \frac{5}{2} x_2 + \frac{3}{2} x_3$

$w_1 = 12 - (3 - \frac{1}{2} w_2 - \frac{1}{2} x_2 + \frac{1}{2} x_3) - 2x_2 - x_3 = 9 + \frac{1}{2} w_2 - \frac{3}{2} x_2 - \frac{3}{2} x_3$ $1/6$, biggest

$x_1 = \quad\quad\quad\quad\quad\quad = 3 - \frac{1}{2} w_2 - \frac{1}{2} x_2 + \frac{1}{2} x_3$ $-1/6$

$w_3 = 9 + (3 - \frac{1}{2} w_2 - \frac{1}{2} x_2 + \frac{1}{2} x_3) - 3x_2 \quad = 12 - \frac{1}{2} w_2 - \frac{7}{2} x_2 + \frac{1}{2} x_3$ $-1/24$

x_3 enters, w_1 leaves: $w_1 = 9 + \frac{1}{2} w_2 - \frac{3}{2} x_2 - \frac{3}{2} x_3 \Rightarrow x_3 = 6 + \frac{1}{3} w_2 - x_2 - \frac{2}{3} w_1$

$\xi = 3 - \frac{1}{2} w_2 - \frac{5}{2} x_2 + \frac{3}{2}(6 + \frac{1}{3} w_2 - x_2 - \frac{2}{3} w_1) = 12 - 4x_2 - w_1$ optimal!

$x_3 \quad\quad\quad\quad\quad\quad = 6 + \frac{1}{3} w_2 - x_2 - \frac{2}{3} w_1$

$x_1 = 3 - \frac{1}{2} w_2 - \frac{1}{2} x_2 + \frac{1}{2}(6 + \frac{1}{3} w_2 - x_2 - \frac{2}{3} w_1) = 6 - \frac{1}{3} w_2 - x_2 - \frac{1}{3} w_1$

$w_3 = 12 - \frac{1}{2} w_2 - \frac{7}{2} x_2 + \frac{1}{2}(6 + \frac{1}{3} w_2 - x_2 - \frac{2}{3} w_1)$

$\frac{1}{3}w_1) = 15 - \frac{1}{3}w_2 - 4x_2 - \frac{1}{3}w_1$

12 is optimal value, for $x_2 = w_1 = 0$

we get this when $x_2 = w_1 = w_2 = 0$, and $x_3 = 6, x_1 = 6, w_3 = 15$

Note: We can increase w_2 in the objective, $\vec{x} = (x_1, x_2, x_3) = (6, 0, 6)$ and still be optimal.

We require:

1. eq: $6 + \frac{1}{3}w_2 \geq 0 \Rightarrow w_2 \geq -18$ always ok
2. eq.: $6 - \frac{1}{3}w_2 \geq 0 \Rightarrow w_2 \leq 18$
3. eq: $15 - \frac{1}{3}w_2 \geq 0 \Rightarrow w_2 \leq 45$

} Must have $0 \leq w_2 \leq 18$

General solution: $\vec{x} = (x_1, x_2, x_3) = (6 - \frac{1}{3}w_2, 0, 6 + \frac{1}{3}w_2)$, $0 \leq w_2 \leq 18$