Lecture 10/2

Feasible dictionary in chap. 2: One where $\bar{b}_i \geq 0$

Simplex method preserves the property $\bar{b}_i \geq 0$

What happens when $\bar{b}_i = 0$. Dictionary is then called degenerate (otherwise nondegenerate)

Let the feasible region be on the form $Ax \leq \vec{b}$

$$\max \frac{\bar{a}_{ik}}{\bar{b}_i}$$

- ≤ 0. The problem is unbounded $x_k \to \infty$
- $0 < \cdot < \infty$ some $\bar{b}_i \geq 0$, $\bar{a}_{ik} > 0$ can increase x_k, but not to ∞
- ∞ : $\bar{a}_{ik} > 0$, $\bar{b}_i = 0$ for some $i \in B$

Can't increase x_k, but can still perform the pivot. Such a pivot is called degenerate.

Note: As long as pivots are nondegenerate, the objective value will improve
as long as objective value improves, simplex will eventually terminate.

What can go wrong: We can cycle between degenerate pivots infinitely many times.

Example where we break away from degeneracy

ξ	$= 3$	$-0.5x_1$	$+2x_2$	$-1.5w_1$	dictionary is degenerate
x_3	$= 1$	$-0.5x_1$		$-0.5w_1$	$\frac{0}{1} = 0$
w_2	$= 0$	x_1	$-x_2$	$+w_1$	$\frac{1}{0} = \infty$, biggest, w_2 leaves

$w_2 = x_1 - x_2 + w_1 \iff x_2 = x_1 - w_2 + w_1$

$$\xi = 3 + 1.5x_1 - 2w_2 + 0.5w_1$$
$$x_3 = 1 - 0.5x_1 - 0.5w_1$$
$$x_2 = 0 + x_1 - w_2 + w_1$$

also degenerate

ratios

0.5, biggest, x_3 leaves

$\frac{-1}{0} = -\infty$

$x_3 = 1 - 0.5x_1 - 0.5w_1 \iff x_1 = 2 - 2x_3 - w_1$

$$\xi = 6 - 3x_3 - 2w_2 - w_1$$
$$x_1 = 2 - 2x_3 - w_1$$
$$x_2 = 2 - 2x_3 - w_2$$

Non degenerate dictionary

Dictionary is also optimal.

6 is optimal value, when $x_3 = w_2 = w_1 = 0$

$x_1 = x_2 = 2$

$\vec{x} = (x_1, x_2, x_3) = (2, 2, 0)$

Example where cycling occurs

$$\xi = x_1 - 2x_2 - 2x_4$$
$$w_1 = -0.5x_1 + 3.5x_2 + 2x_3 - 4x_4$$
$$w_2 = -0.5x_1 + x_2 + 0.5x_3 - 0.5x_4$$
$$w_3 = 1 - x_1$$

x_1 enters

ratios

$\frac{0.5}{0} = \infty$, choose as leaving

$\frac{0.5}{0} = \infty$

1

$$x_1 = -2w_1 + 7x_2 + 4x_3 - 8x_4$$

$$\xi = -2w_1 + 5x_2 + 4x_3 - 10x_4$$
$$x_1 = -2w_1 + 7x_2 + 4x_3 - 8x_4$$
$$w_2 = w_1 - 2.5x_2 - 1.5x_3 + 3.5x_4$$
$$w_3 = 1 + 2w_1 - 7x_2 - 4x_3 + 8x_4$$

x_2 enters

ratios:

$\frac{-7}{0} = -\infty$

$\frac{2.5}{0} = \infty$, biggest, w_2 leaves.

$\frac{7}{1} = 7$

..... 6 more pivots, and you get back to the same dictionary !

Remedies for cycling:

The lexiographic method perturbs the $\bar{b}_i$ at different scales.

$$\xi = 6x_1 - 4x_2$$
$$w_1 = 9x_1 + 4x_2$$
$$w_2 = -4x_1 - 2x_2$$
$$w_3 = 1 - x_2$$

nondegenerate

$$\xi = +6x_1 - 4x_2$$
$$w_1 = \varepsilon_1 + 9x_1 + 4x_2$$
$$w_2 = \varepsilon_2 - 4x_1 - 2x_2$$
$$w_3 = 1 + \varepsilon_3 - x_2$$

x_1 enters

ratios:

$-\frac{9}{\varepsilon_1}$

$\frac{4}{\varepsilon_2}$, biggest, w_2 leaves.

$\frac{0}{1+\varepsilon_3} = 0$

$0 < \varepsilon_3 << \varepsilon_2 << \varepsilon_1$

"much" smaller than.

$x_1 = \frac{1}{4}\varepsilon_2 - \frac{1}{4}w_2 - \frac{1}{2}x_2$

After this + one more pivot:

$$\begin{aligned} \xi &= 2\varepsilon_2 - 2w_2 - 2x_1 \\ w_1 &= \varepsilon_1 + 2\varepsilon_2 - 2w_2 + x_1 \\ x_2 &= 0.5\varepsilon_2 - 0.5w_2 - 2x_1 \\ w_3 &= 1 - 0.5\varepsilon_2 + \varepsilon_3 + 0.5w_2 + 2x_1 \end{aligned}$$

perturbed

$$\begin{aligned} \xi &= -2w_2 - 2x_1 \\ w_1 &= -2w_2 + x_1 \\ x_2 &= -0.5w_2 - 2x_1 \\ w_3 &= 1 + 0.5w_2 + 2x_1 \end{aligned}$$

unperturbed, optimal.

"ε-matrix": $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0.5 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}$

Theorem 3.2: Simplex always terminates when leaving variable is chosen according to the lexicographic rule

Proof: "ε-matrix" has full rank at all steps.

$\Downarrow$

all dictionaries will be nondegenerate.

$\Downarrow$

Simplex will eventually terminate.

Finally, set $\varepsilon_i = 0$ ∎

Remedies for cycling 2

Blands rule: Choose entering/leaving variables x_k from the choice set as the one with the smallest index (the k in x_k)

Theorem 3.4 Fundamental theorem of LP.

For an arbitrary linear program:

1. If there is no optimal solution, the problem is either unbounded or infeasible.
2. If a feasible solution exists, a basic feasible optimal solution also exists.