Chap.4 : Efficiency of simplex algorithm

We have addressed the following:

1. Feasibility (simplex phase 1)
2. Optimality (simplex phase 2)
3. Unboundedness (ratios ≤ 0)
4. Degeneracy, cycling (Blands rule, Lexicographic.

Question: How efficient is simplex?

Measures for efficiency:

1. Worst-case: maximum number of arithmetic operations for problem instances of a given "size"
2. Mean/average: Expected number of arithmetic operations.

Measures for problem size

1. Number of elements in the input LP $(n+1) \times (m+1)$ = size of dict.
2. Number of nonzeros in the input.
3. Number of bits needed to store all input elements.

Measures for the total work of an algorithm: (complexity)

1. Computational time
2. Number of iterations.
3. Number of arithmetic operations.

We will use m and n as problem size measure, and the number of pivots as measure for complexity.

Worst-case analysis of simplex

Simplex is not good in a worst-case scenario.
It still works well in practice, since the "hard" LP problems are rare in practice.

The Klee-Minty problem: has n variables and n constraints

Turns out that number of pivots needed, when using the largest coefficient rule, is $2^n - 1$.

Note: there are $\binom{2n}{n}$ possible dictionaries.

In exercise 4.9 we show that $\frac{1}{2n} 2^{2n} \leq \binom{2n}{n} \leq 2^{2n}$ i.e.

number of pivots in Klee Minty $\approx$ square root of total number of possible dictionary. $2^n = \sqrt{2^{2n}} \approx \sqrt{\binom{2n}{n}}$

Proof: $2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} 1^k 1^{2n-k} = \sum_{k=0}^{2n} \binom{2n}{k}$

$$\binom{2n}{n} \leq \sum_{k=0}^{2n} \binom{2n}{k} \leq (2n+1)\binom{2n}{n} \leftarrow \text{biggest.}$$

$$\binom{2n}{n} \leq 2^{2n} \leq (2n+1)\binom{2n}{n}$$

$$\Downarrow$$

$$\frac{1}{2n+1} 2^{2n} \leq \binom{2n}{n} \leq 2^{2n}$$

Idea behind Klee-Minty: The unit cube in $\mathbb{R}^n$ ($[0,1]^n$) has 2^n vertices. Deform this cube to force simplex to visit all vertices.

Klee Minty formulation:

$$\max \sum_{j=1}^{n} 2^{n-j} x_j$$

$$\text{s.t. } 2\sum_{j=1}^{i-1} 2^{i-j} x_j + x_i \leq 100^{i-1} \qquad i = 1, \ldots, n$$

$$x_j \geq 0 \qquad j = 1, \ldots n$$

$$\max \quad 2^{n-1}x_1 + 2^{n-2}x_2 + 2^{n-3}x_3 + 2^{n-4}x_4 + \cdots + 8x_{n-3} + 4x_{n-2} + 2x_{n-1} + x_n$$

$$\begin{aligned}
x_1 &\le 1 \\
4x_1 + x_2 &\le 100 \\
8x_1 + 4x_2 + x_3 &\le 100^2 \\
16x_1 + 8x_2 + 4x_3 + x_4 &\le 100^3 \\
&\vdots
\end{aligned}$$

In particular $x_i \le 100^{i-1}$. We will use a modified version where

1. Right hand sides: $100^{i-1} \to B_i$ with $1 = B_1 << B_2 << \cdots << B_n$
2. $B_i \to \sum_{j=1}^{i-1} 2^{i-j} B_j + B_i$ (small perturbation of B_i)
3. Add a constant to the objective.

Modified Klee-Minty:

$$\max \quad \sum_{j=1}^{n} 2^{n-j} x_j - \frac{1}{2} \sum_{j=1}^{n} 2^{n-j} B_j$$

$$\text{s.t.} \quad 2 \sum_{j=1}^{i-1} 2^{n-j} x_j + x_i \le \sum_{j=1}^{i-1} 2^{i-j} B_j + B_i$$

Exercise 4.7 proves this problem requires $2^n - 1$ pivots.

For $n = 3$:

$$\begin{aligned}
\xi &= -2B_1 - B_2 - \tfrac{1}{2}B_3 + 4x_1 + 2x_2 + x_3 \\
w_1 &= B_1 - x_1 \\
w_2 &= 2B_1 + B_2 - 4x_1 - x_2 \\
w_3 &= 4B_1 + 2B_2 + B_3 - 8x_1 - 4x_2 - x_3
\end{aligned}$$

x_1 enters

ratios

$1/B_1$ biggest, w_1 leaves

$4/(2B_1 + B_2)$

$8/(4B_1 + 2B_2 + B_3)$

$w_1 = B_1 - x_1$, $x_1 = B_1 - w_1$

$$\begin{aligned}
\xi &= 2B_1 - B_2 - \tfrac{1}{2}B_3 - 4w_1 + 2x_2 + x_3 \\
x_1 &= B_1 - w_1 \\
w_2 &= -2B_1 + B_2 + 4w_1 - x_2 \\
w_3 &= -4B_1 + 2B_2 + B_3 + 8w_1 - 4x_2 - x_3
\end{aligned}$$

x_2 enters

ratios:

0 biggest, w_2 leaves

$1/(-2B_1 + B_2)$

$4/(-4B_1 + 2B_2 + B_3)$

$$x_2 = -2B_1 + B_2 + 4w_1 - w_2$$

$$\begin{aligned}
\xi &= -2\beta_1 + \beta_2 - \tfrac{1}{2}\beta_3 + 4w_1 - 2w_2 + x_3 \\
x_1 &= \beta_1 - w_1 \\
x_2 &= -2\beta_1 + \beta_2 + 4w_1 - w_2 \\
w_3 &= 4\beta_1 - 2\beta_2 + \beta_3 - 8w_1 + 4w_2 - x_3
\end{aligned}$$

w_1 enters
x_1 leaves

$$\begin{aligned}
\xi &= 2\beta_1 + \beta_2 - \tfrac{1}{2}\beta_3 - 4x_1 - 2w_2 + x_3 \\
w_1 &= \beta_1 - x_1 \\
x_2 &= 2\beta_1 + \beta_2 - 4x_1 - w_2 \\
w_3 &= -4\beta_1 - 2\beta_2 + \beta_3 + 8x_1 + 4w_2 - w_3
\end{aligned}$$

x_3 enters
w_3 leaves

$$\begin{aligned}
\xi &= -2\beta_1 - \beta_2 + \tfrac{1}{2}\beta_3 + 4x_1 + 2w_2 - w_3 \\
w_1 &= \beta_1 - x_1 \\
x_2 &= 2\beta_1 + \beta_2 - 4x_1 - w_2 \\
x_3 &= -4\beta_1 - 2\beta_2 + \beta_3 + 8x_1 + 4w_2 - w_3
\end{aligned}$$

Rules: When w_i or x_i enters:

1. $\beta_1, \dots, \beta_i$ $w_1 \cdots w_i$ $x_1 \cdots x_i$ change sign in the objective.
2. In the constraints: "β-matrix": stay lower triangular, "x-matrix": with 1 on the diagonal.
3. i'th constraint leaves when i'th variable enters, and vice versa.

Coefficients in the matrix change as follows:

$$(4,2,1) \to (-4,2,1) \to (4,-2,1) \to (4,2,1) \to (4,2,-1)$$

$$\to (-4,2,-1) \to (4,-2,-1) \to (-4,-2,-1)$$

We needed $7 = 2^3 - 1$ pivots, proving Klee-Minty for $n=3$.

General proof by induction:

Assume the $(n-1)$-case requires $2^{n-1}-1$ pivots. i.e.,

$$(2^{n-2}, 2^{n-3}, \dots, 1) \xrightarrow{2^{n-1}-1 \text{ pivots}} (-2^{n-2}, -2^{n-3}, \dots, -1)$$

The n-case:

$$(2^{n-1}, 2^{n-2}, \ldots, 1) \longrightarrow (-2^{n-1}, -2^{n-2}, \ldots, -1)$$

c changes sign in all pivots.
when + in front, only first coefficient changes sign.
when − in front: sign change in other coeffs as well.

Total number of pivots:

$$2(2^{n-1}-1)+1 = 2^n - 2 + 1 = 2^n - 1$$

Remarks:

1. Unknown whether there exists a pivoting rule that can avoid 2^n pivots for all input.
2. Expected number of pivots needed for a randomly generated problem is $n^3 m^{\frac{1}{n+1}}$
3. In practice between m and $2m$ pivots necessary