Lecture 17/2

Exercise 2.16

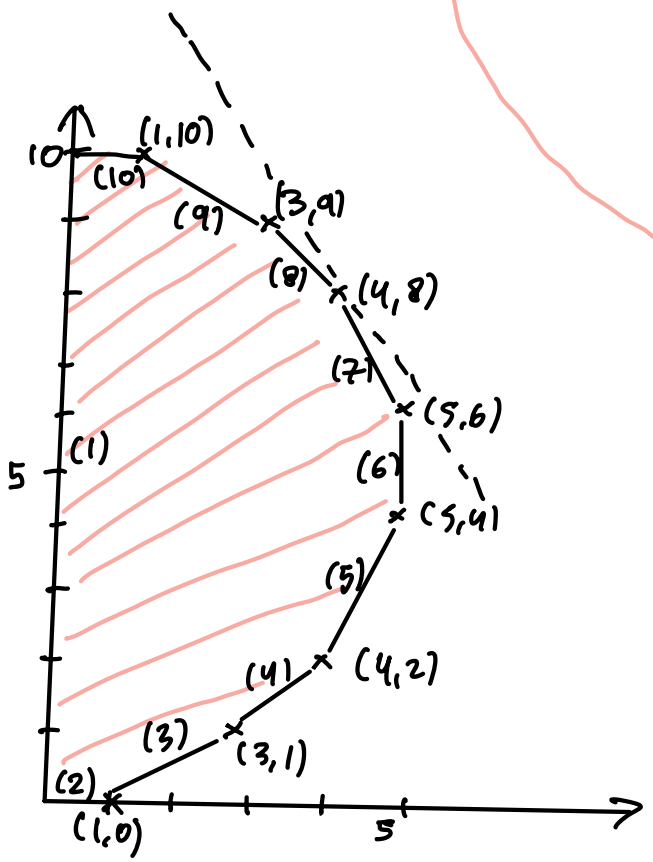
Objective: $3x_1 + 2x_2$

Contours: $3x_1 + 2x_2 = c$

$x_2 = -\frac{3}{2}x_1 + c$

$k = -\frac{1}{2}$, between 1 and 2

constraint	slack	k
(1): $x_2 \geq 0$		
(2): $x_1 \geq 0$		
(3): $x_1 - 2x_2 \leq 1$	w_1	$\frac{1}{2}$
(4): $x_1 - x_2 \leq 2$	w_2	1
(5): $2x_1 - x_2 \leq 6$	w_3	2
(6): $x_1 \leq 5$	w_4	∞
(7): $2x_1 + x_2 \leq 16$	w_5	-2
(8): $x_1 + x_2 \leq 12$	w_6	-1
(9): $x_1 + 2x_2 \leq 21$	w_7	$-\frac{1}{2}$
(10): $x_2 \leq 10$	w_8	0



max intersection between (7) and (8)

$\Downarrow$

$(4,8)$

$f(4,8) = 3 \cdot 4 + 2 \cdot 8 = 28$

LCR starts with x_1 entering

x_1 increases: $(0,0) \rightarrow (1,0)$

path taken

$(0,0) \rightarrow (1,0) \rightarrow (3,1) \rightarrow (4,2) \rightarrow (5,4)$

$0 \quad 3 \quad 11 \quad 12 \quad 23$

$\rightarrow (5,6) \rightarrow (4,8)$

$27 \quad 28$

Ex. 2.18 Show that, if X_k leaves, it does not enter in the next iteration.

Solution: $\xi = c_1 X_1 + \cdots + c_r X_r + \cdots + c_n X_n$

Assume X_r enters, X_k leaves

constraint k:

$$X_k = b_k - \sum_{j \neq r} a_{kj} X_j - a_{kr} X_r$$

X_r and X_k swap

$$X_r = \frac{b_k}{a_{kr}} - \frac{1}{a_{kr}} X_k - \cdots$$

can assume $a_{kr} > 0$, else the problem is unbounded.

$$c_r X_r = c_r \left(\frac{b_k}{a_{kr}} - \frac{1}{a_{kr}} X_k - \cdots \right)$$

$$= \frac{c_r b_k}{a_{kr}} - \frac{c_r}{a_{kr}} X_k - \cdots$$

negative, since $c_r > 0$, $a_{kr} > 0$ (X_r entered)

Thus: coefficient of X_k in the objective is < 0 in next iteration, so is not a candidate for entering in next iteration.

Ex. 2.19

$$\max \sum_{j=1}^{n} p_j X_j$$

$$\text{s.t.} \sum_{j=1}^{n} q_j X_j \leq \beta$$

$$0 \leq X_j \leq 1$$

assumptions: positive

$$\sum_{j=1}^{n} p_j = 1 = \sum_{j=1}^{n} q_j$$

$$\frac{p_1}{q_1} < \cdots < \frac{p_n}{q_n}$$

β small, positive.

Solving without simplex : substitute $y_j = q_j x_j$

$$\max \sum_{j=1}^{n} \frac{p_j}{q_j} y_j$$

$$\text{s.t.} \sum_{j=1}^{n} y_j \le B$$

$$0 \le y_j \le q_j$$

by assumption that B is small.

We must clearly set $y_n = \min(B, q_n) = B$

$$\Rightarrow x_n = \frac{y_n}{q_n} = \frac{B}{q_n}$$

Also $y_1 = \cdots = y_{n-1} = 0$

$x_1 = \cdots = x_{n-1} = 0$ $\Rightarrow \vec{x} = (0, \ldots, 0, \frac{B}{q_n})$

optimal value : $p_n x_n = B \frac{p_n}{q_n}$

Solving with simplex

$$\begin{aligned}
\xi &= p_1 x_1 + \cdots + p_n x_n \\
w_1 &= 1 - x_1 \\
&\vdots \\
w_n &= 1 - x_n \\
w_{n+1} &= B - q_1 x_1 - \cdots - q_n x_n
\end{aligned}$$

ratios

$0/1 = 0$

$1/1 = 1$

$\frac{q_n}{B} >> 1$, biggest, w_{n+1} leaves

$$x_n = \frac{B}{q_n} - \frac{q_1}{q_n} x_1 - \cdots - \frac{q_{n-1}}{q_n} x_{n-1} - \frac{1}{q_n} w_{n+1}$$

x_n can be increased to $\frac{B}{q_n}$

$$\xi = p_1 x_1 + \cdots + p_{n-1} x_{n-1} + p_n \left(\frac{B}{q_n} - \frac{q_1}{q_n} x_1 - \cdots - \frac{q_{n-1}}{q_n} x_{n-1} - \frac{1}{q_n} w_{n+1} \right)$$

$$= \frac{p_n}{q_n} B + \left(p_1 - p_n \frac{q_1}{q_n}\right) x_1 + \cdots + \left(p_{n-1} - p_n \frac{q_{n-1}}{q_n}\right) x_{n-1} - \frac{p_n}{q_n} w_{n+1}$$

$$= \frac{p_n}{q_n} B + q_1 \left(\frac{p_1}{q_1} - \frac{p_n}{q_n}\right) x_1 + \cdots + q_{n-1} \left(\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{x_n}\right) x_n - \frac{p_n}{q_n} w_{n+1}$$

<0, $\frac{p_n}{q_n}$ is biggest <0 <0

$\Rightarrow$ we get an optimal dictionary with $\frac{p_n}{q_n}\beta$ as optimal value

Also: $x_1 = \dots x_{n-1} = 0$ (nonbasic), $x_n = \frac{\beta}{q_n}$

$\Rightarrow \vec{x} = (x_1, \dots, x_n) = (0, \dots, 0, \frac{\beta}{q_n})$.

Ex 3.1

$$\begin{aligned} \max \quad & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{s.t.} \quad & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \le 0 \\ & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \le 0 \\ & x_1 \le 1 \end{aligned}$$

Solve this using lexiographic rule ($\varepsilon_1 >> \varepsilon_2 >> \varepsilon_3$) perturbed problem:

$$\begin{aligned} \xi &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ w_1 &= \varepsilon_1 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ w_2 &= \varepsilon_2 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ w_3 &= 1 + \varepsilon_3 - x_1 \end{aligned}$$

x_1 enters ratios: $0.5/\varepsilon_1$, $0.5/\varepsilon_2$, $1/(1+\varepsilon_3)$

biggest, w_2 leaves

$$x_1 = 2\varepsilon_2 - 2w_2 + 3x_2 + x_3 - 2x_4$$

$$\begin{aligned} \xi &= 20\varepsilon_2 - 20w_2 - 27x_2 + x_3 - 44x_4 \\ w_1 &= \varepsilon_1 - \varepsilon_2 + w_2 + 4x_2 + 2x_3 - 8x_4 \\ x_1 &= 2\varepsilon_2 - 2w_2 + 3x_2 + x_3 - 2x_4 \\ w_3 &= 1 - 2\varepsilon_2 + \varepsilon_3 + 2w_2 - 3x_2 - x_3 + 2x_4 \end{aligned}$$

x_3 enters ratios: $-2/(\varepsilon_1 - \varepsilon_2)$, $-1/(2\varepsilon_2)$, $1/(1 - 2\varepsilon_2 + \varepsilon_3)$

biggest, so w_3 leaves

$$\begin{aligned} \xi &= 1 + 18\varepsilon_2 + \varepsilon_3 - 18w_2 - 30x_2 - w_3 - 42x_4 \\ w_1 &= 2 + \varepsilon_1 - 5\varepsilon_2 + 2\varepsilon_3 + 5w_2 - 2x_2 - 2w_3 - 4x_4 \\ x_1 &= 1 + \varepsilon_3 - w_3 \\ x_3 &= 1 - 2\varepsilon_2 + \varepsilon_3 + 2w_2 - 3x_2 - w_3 + 2x_4 \end{aligned}$$

Optimal with value 1 (set $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$)

$w_2 = x_2 = w_3 = x_4 = 0$, $x_1 = 1$, $x_3 = 1$

$\Rightarrow \vec{x} = (x_1, x_2, x_3, x_4) = (1, 0, 1, 0)$

Ex. 3.2 Solving the same system with Bland's rule:

$$\begin{aligned} \xi &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ w_1 &= -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ w_2 &= -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ w_3 &= 1 - x_1 \end{aligned}$$

x_1 enters

ratios

$0.5/0 = \infty$ } biggest

$0.5/0 = \infty$ } w_1 leaves, the smallest index.

$1/1 = 1$

$$\begin{aligned} \xi &= -20w_1 + 53x_2 + 41x_3 - 204x_4 \\ x_1 &= -2w_1 + 11x_2 + 5x_3 - 18x_4 \\ w_2 &= w_1 - 4x_2 - 2x_3 + 8x_4 \\ w_3 &= 1 + 2w_1 - 11x_2 - 5x_3 + 18x_4 \end{aligned}$$

x_2 enters (smallest index)

ratios:

$-11/0 = -\infty$

$4/0 = \infty$ biggest, w_2 leaves

$11/1 = 11$

$$\begin{aligned} \xi &= -\tfrac{27}{4}w_1 - \tfrac{53}{4}w_2 + \tfrac{29}{2}x_3 - 98x_4 \\ x_1 &= \tfrac{3}{4}w_1 - \tfrac{11}{4}w_2 - \tfrac{1}{2}x_3 + 4x_4 \\ x_2 &= \tfrac{1}{4}w_1 - \tfrac{1}{4}w_2 - \tfrac{1}{2}x_3 + 2x_4 \\ w_3 &= 1 - \tfrac{3}{4}w_1 + \tfrac{11}{4}w_2 + \tfrac{1}{2}x_3 - 4x_4 \end{aligned}$$

x_3 enters

ratios

$0.5/0 = \infty$ } x_1 leaves, smallest index.

$0.5/0 = \infty$ }

$\frac{-0.5}{1} = -0.5$

Note: No improvement in the objective so far.

on PC: No improvement in the next 4 iterations either, but then the program jumps directly to the optimal value.

Ex. 3.4

$$\max \sum_j c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le 0 \qquad i = 1, \dots, m$$

$$x_j \ge 0 \qquad j = 1, \dots, n$$

will show: either $\vec{x} = \vec{0}$ is optimal, or the problem is unbounded

Assume that $c_j \leq 0$ for all j. Then dictionary is optimal, and $\vec{x} = \vec{0}$ is optimal

Assume that some $c_k > 0$, and let x_k enter:

$$\xi = c_1 x_1 + \cdots + c_k x_k + \cdots + c_n x_n$$
$$w_1 = -a_{11} x_1 - \cdots - a_{1k} x_k - \cdots - a_{1n} x_n$$
$$\vdots$$
$$w_m = -a_{m1} x_1 - \cdots - a_{mk} x_k - \cdots - a_{mn} x_n$$

ratios:

$a_{1k}/0 = 0$ or $\pm\infty$

$a_{mk}/0 = 0$ or $\pm\infty$

if all ratios ≤ 0, the problem is unbounded.

if some ratio is ∞, say $\frac{a_{ck}}{0} = \infty$, then $a_{ck} > 0$

constraint c then says $0 - a_{ck} x_k = 0 \Rightarrow x_k = 0$

This shows that you can't increase the entering variable, so after all pivots, $\vec{x}$ remains at $\vec{0}$, so $\vec{0}$ is optimal.