Lecture 18/2.

$\frac{0}{0}$ is defined as 0 : $b_i + a_{ik} x_k = 0$, $b_i = 0$, $a_{ik} = 0$

can increase x_k as much as you want.

To every LP problem (P), there is another LP problem (D), called the dual problem. (P) is called the primal problem

(P)

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, \dots, m$$

$$x_j \ge 0 \quad j = 1, \dots, n$$

(D)

$$\min \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} y_i a_{ij} \ge c_j \quad j = 1, \dots, n$$

$$y_i \ge 0 \quad i = 1, \dots, m$$

Why consider (D)? can give quick bounds for optimal value of (P)
one can solve (P) by solving (D)
and solving (D) may be more efficient.

Transformation from (P) to (D):

- For (P): rows of A $\leftrightarrow$ constraints
  columns of A $\leftrightarrow$ variables
- For (D): columns of A $\leftrightarrow$ constraints
  rows of A $\leftrightarrow$ variables
- b_i are right hand sides in (P), but in the objective in (D)
- c_j are in the objective in (P), but in the right hand sides in the constraints in (D)
- constraint inequalities are reversed
- max is replaced by min.

Theorem 5.1 (Weak duality)

Let $(x_1, \dots, x_n)$ be feasible for (P), and $(y_1, \dots, y_m)$ be feasible for (D).

Then $\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i$

Proof: $\sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij}\right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i \le \sum_{i=1}^{m} b_i y_i$ ■

Example

(P)

$$\begin{aligned} \max \quad & 2x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 1 \\ & 2x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

(D)

$$\begin{aligned} \min \quad & y_1 + y_2 \\ \text{s.t.} \quad & y_1 + 2y_2 \ge 2 \\ & 2y_1 + y_2 \ge 2 \\ & y_1, y_2 \ge 0 \end{aligned}$$

$(\frac{1}{4}, \frac{1}{4})$ feasible for (P), value 1 $(1,1)$ feasible for (D), value 2

weak duality: optimal values for both (P) and (D) must lie in $[1, 2]$.

$(\frac{1}{3}, \frac{1}{3})$ feasible for (P), value $\frac{4}{3}$ $(\frac{2}{3}, \frac{2}{3})$ feasible for (D), value $\frac{4}{3}$.

Weak duality: optimal value for both (P) and (D) must be $\frac{4}{3}$

Interpretation of (D):

"Find a combination of the constraints of (P)

$$\sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j\right) \le \sum_{i=1}^{m} b_i y_i$$

with the right hand side as small as possible, and so that the coefficient "overshoot" c_j:

$\sum y_i a_{ij} \ge c_i$ } This gives something $\ge \max$ (P).

"Find the best way to combine constraints in (P), so that the combined coefficients dominate those in the objective function".

Theorem 5.2 (Strong duality)

If (P) has an optimal solution $x^* = (x_1^*, \dots, x_n^*)$, then (D) also has an optimal solution $y^* = (y_1^*, \dots, y_m^*)$, so that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Idea behind proof: The simplex method solves (P) and (D) simultaneously, with same optimal value.

(P)

$$\max \ \vec{c}^T \vec{x} \quad \text{s.t.} \ A\vec{x} \le \vec{b}, \ \vec{x} \ge \vec{0}$$

— standard forms —

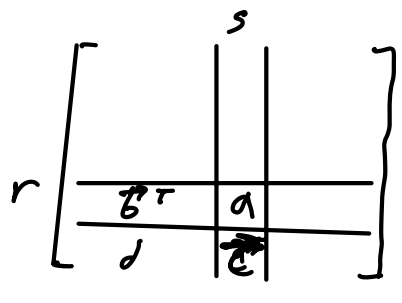
$$\max \ -\vec{b}^T \vec{y} \quad \text{s.t.} \ -A^T \vec{y} \le -\vec{c}, \ \vec{y} \ge \vec{0}$$

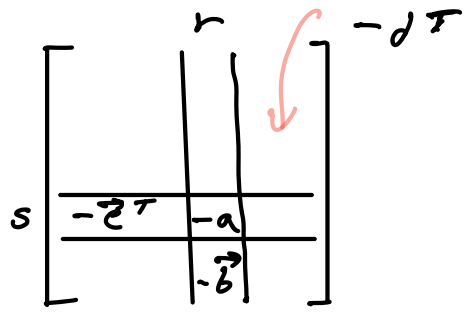
$$\begin{bmatrix} 0 & \vec{c}^T \\ \vec{b} & -A \end{bmatrix} \quad \text{— Dictionaries —} \quad \begin{bmatrix} 0 & -\vec{b}^T \\ -\vec{c} & A^T \end{bmatrix}$$

These are negative transposes

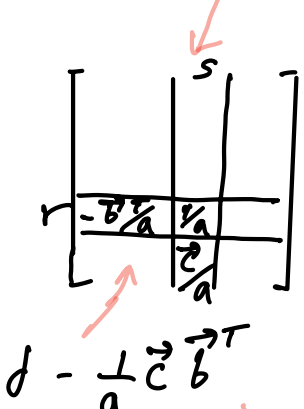
Claim: If we perform a (r,s)-pivot to (P), (leaving, entering) and the (s,r) pivot to (D), the resulting dictionaries will still be negative transposes of one-another.

Proof:





pivot.m : $D \to D^{new}$:



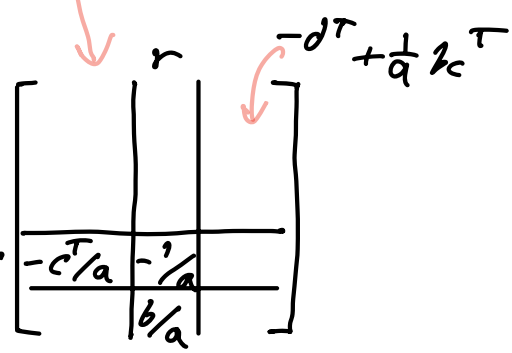
$$D^{new} = D - \frac{1}{D_{rs}} D_{:,s} D_{r,:}$$

$$D^{new}_{:,s} = \frac{1}{D_{rs}} D_{:,s}$$

$$D^{new}_{r,:} = -\frac{1}{D_{rs}} D_{r,:}$$

$$D^{new}_{r,s} = \frac{1}{D_{rs}}$$

$$A - \frac{1}{a}\vec{c}\,\vec{b}^T$$



These are still negative transposes of one another

Example:

(P)

$$\eta = 4x_1 + x_2 + 3x_3$$
$$w_1 = 1 - x_1 + 4x_2$$
$$w_2 = 3 - 3x_1 + x_2 - x_3$$

ratios: 0, $\frac{1}{3}$ biggest

w_2 leave

(D)

$$-\xi = -y_1 - 3y_2$$
$$z_1 = -4 + y_1 + 3y_2$$
$$z_2 = -1 - 4y_1 - y_2$$
$$z_3 = -3 + y_2$$

$x_3 = 3 - 3x_1 + x_2 - w_2$

$$\eta = 9 - 5x_1 + 4x_2 - 3w_2$$
$$w_1 = 1 - x_1 - 4x_2$$
$$x_3 = 3 - 3x_1 + x_2 - w_2$$

$$-\xi = -9 - y_1 - 3z_3$$
$$z_1 = 5 + y_1 + 3z_3$$
$$z_2 = -4 + 4y_1 - z_3$$
$$y_2 = 3 + z_3$$

x_2 enters. ratios are $4/1 = 4$, biggest, w_1 leaves

$-\frac{1}{3}$

$$\eta = 10 - 6x_1 - w_1 - 3w_2$$
$$x_2 = \frac{1}{4} - \frac{1}{4}x_1 - \frac{1}{4}w_1$$
$$x_3 = \frac{13}{4} - \frac{13}{4}x_1 - \frac{1}{4}w_1 - w_2$$

$$-\xi = -10 - \frac{1}{4}z_2 - \frac{13}{4}z_3$$
$$z_1 = 6 + \frac{1}{4}z_2 + \frac{13}{4}z_3$$
$$y_1 = 1 + \frac{1}{4}z_2 + \frac{1}{4}z_3$$
$$y_2 = 3 + 0 \cdot z_2 + 1 \cdot z_3$$

Both dictionaries are feasible and optimal!

Optimal values: $\eta = 10$ $\quad -\xi = -10$, $\xi = 10$

$\vec{x} = (x_1, x_2, x_3) = (0, \frac{1}{4}, \frac{13}{4})$ $\quad \vec{y} = (y_1, y_2) = (1, 3)$

$\vec{w} = (w_1, w_2) = (0, 0)$ $\quad \vec{z} = (z_1, z_2, z_3) = (6, 0, 0)$

Mapping between variables: $x_j \leftrightarrow z_j$

$w_i \leftrightarrow y_i$

In general: (P) is optimal $\Rightarrow$ (D) is feasible

(P) is feasible $\Rightarrow$ (D) is optimal

It follows: (P) optimal and feasible $\Leftrightarrow$ (D) is optimal and feasible.