Lecture 24/2

Direct proof of strong duality

It is enough to find a dual feasible y^* so that

$$(*) \quad \sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Assume that the simplex method produces the optimal solution x^* for (P):

Final dictionary:

$$\xi = \xi^* + \sum_{j \in \mathcal{N}} c_j^* x_j = \xi^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=1}^{m} d_i^* w_i$$

split variables into slack/nonslack, add 0-coeffs.

$(*1)$: $\xi^* + \sum_{j=1}^{n} c_j x_j^*$ (optimum inserted in starting dictionary)

Define $y_i^* = -d_i^*$. We get

$(*2)$: $y_i^* \geq 0$ (since the final dictionary is optimal)

Also:

$$\sum_{j=1}^{n} c_j x_j = \xi^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=1}^{m} d_i^* w_i$$

$$= \xi^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=1}^{m} (-y_i^*)\left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right)$$

$$= \xi^* - \sum_{i=1}^{m} b_i y_i^* + \sum_{j=1}^{n} \left(c_j^* + \sum_{i=1}^{m} y_i^* a_{ij}\right) x_j$$

Compare left/right: $\sum_{i=1}^{m} b_i y_i^*$ is the 0; $c_j^* + \sum_{i=1}^{m} y_i^* a_{ij}$ is c_j.

$(*3)$: $\xi^* = \sum_{i=1}^{m} b_i y_i^*$

$(*4)$: $c_j = c_j^* + \sum_{i=1}^{m} y_i^* a_{ij}$

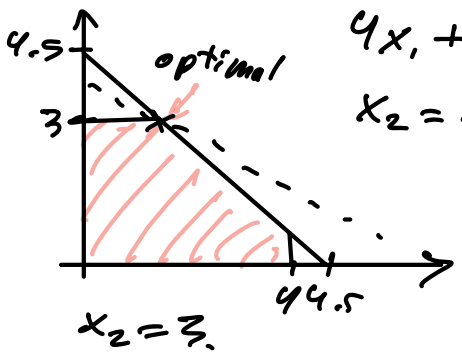
$(*1)$ and $(*3)$ together imply $(*)$

Since $c_j^* \leq 0$ (optimality), $(*4)$ gives

$(*5)$: $c_j \leq \sum_{i=1}^{m} y_i^* a_{ij}$

Because of $(*2)$ and $(*5)$, we have dual feasibility.

Exercise 4.1



$4x_1 + 5x_2 = \text{const}$

$x_2 = -0.8x_1 + \text{const}$

LCR

$\xi = 4x_1 + 5x_2$
$w_1 = 9 - 2x_1 - 2x_2$
$w_2 = 4 - x_1$
$w_3 = 3 - x_2$

x_2 enters
ratios
2/9
0
1/3, biggest

$\xi = 15 + 4x_1 - 5w_3$
$w_1 = 3 - 2x_1 + 2w_3$
$w_2 = 4 - x_1$
$x_2 = 3 - w_3$

x_1 enters
ratios
2/3
1/4
0

2/3 biggest, w_1 leaves

$\xi = 21 - 2w_1 - w_3$
$x_1 = \frac{3}{2} + \dots$
$w_2 = \frac{5}{2} + \dots$
$x_2 = 3 + \dots$

optimal, value 21

$w_1 = w_3 = 0$, $w_2 = \frac{5}{2}$

$x_1 = \frac{3}{2}$, $x_2 = 3$

$\vec{x} = (x_1, x_2) = (\frac{3}{2}, 3)$

Bland's rule

$\xi = 4x_1 + 5x_2$
$w_1 = 9 - 2x_1 - 2x_2$
$w_2 = 4 - x_1$
$w_3 = 3 - x_2$

x_1 enters
ratios
2/9
1/4
0

1/4 biggest, w_2 leaves

$\xi = 16 - 4w_2 + 5x_2$
$w_1 = 1 + 2w_2 - 2x_2$
$x_1 = 4 - w_2$
$w_3 = 3 - x_2$

x_2 enters
ratios
2
0
1/3

2 biggest, w_1 leaves

$\xi = 18.5 + w_2 - 2.5w_1$
$x_2 = 0.5 + w_2 - 0.5w_1$
$x_1 = 4 - w_2$
$w_3 = 2.5 - w_2 + 0.5w_1$

w_2 enters
ratios
-2
1/4
$\frac{1}{2.5} = \frac{2}{5}$

2/5 biggest, w_3 leaves

$\xi = 21 - w_3 - 2w_1$
$x_2 = 3 + \dots$
$x_1 = \frac{3}{2} + \dots$
$w_2 = \frac{5}{2} + \dots$

optimal, same value, same $\vec{x}$.

Ex. 4.2

$$\xi = 2x_1 + x_2$$
$$w_1 = 3 - 3x_1 - x_2$$

x_1 enters, both for LCR and Blands.

$$x_1 = 1 - \tfrac{1}{3}w_1 - \tfrac{1}{3}x_2$$

$$\xi = 2 - \tfrac{2}{3}w_1 + \tfrac{1}{3}x_2$$
$$x_1 = 1 - \tfrac{1}{3}w_1 - \tfrac{1}{3}x_2$$

x_2 enters, both for LCR and Blands

$$x_2 = 3 - w_1 - 3x_1$$

$$\xi = 3 - w_1 - x_1$$
$$x_2 = 3 - w_1 - 3x_2$$

optimal, value 3, $\vec{x} = (x_1, x_2) = (0, 3)$

Ex. 4.4 Klee-Minty for $n = 3$:

$$\xi = 4x_1 + 2x_2 + x_3$$
$$w_1 = 1 - x_1$$
$$w_2 = 100 - 4x_1 - x_2$$
$$w_3 = 1000 - 8x_1 - 4x_2 - x_3$$

ratios
1, biggest, w_1 leaves
$\frac{4}{100}$
8/100

$$\xi = 4 - 4w_1 + 2x_2 + x_3$$
$$x_1 = 1 - w_1$$
$$w_2 = 96 + 4w_1 - x_2$$
$$w_3 = 9992 + 8w_1 - 4x_2 - x_3$$

ratios
0
1/96 biggest, w_2 leaves
4/9992

$$\xi = 196 + 4w_1 - 2w_2 + x_3$$
$$x_1 = 1 - w_1$$
$$x_2 = 96 + 4w_1 - w_2$$
$$w_3 = 9608 - 8w_1 + 4w_2 - x_3$$

ratios
1, biggest, x_1 enters
-4/96
8/9608

$\xi = 200 - 4x_1 - 2w_2 + x_3$
$w_1 = 1 - x_1$
$x_2 = 100 - 4x_1 - w_2$
$w_3 = 9600 + 8x_1 + 4w_2 - x_3$

ratios
0
0
1/9600, biggest, w_3 leaves

$\xi = 9800 + 4x_1 + 2w_2 - w_3$
$w_1 = 1 - x_1$
$x_2 = 100 - 4x_1 - w_2$
$x_3 = 9600 + 8x_1 + 4w_2 - w_3$

ratios
1, biggest, w_1 leaves
4/100
-8/9600

$\xi = 9804 - 4w_1 + 2w_2 - w_3$
$x_1 = 1 - w_1$
$x_2 = 96 + 4w_1 - w_2$
$x_3 = 9608 - 8w_1 + 4w_2 - w_3$

ratios
0
1/96 biggest, x_2 leaves
-4/9608

$\xi = 9996 + 4w_1 - 2x_2 - w_3$
$x_1 = 1 - w_1$
$w_2 = 96 + 4w_1 - x_2$
$w_3 = 9992 + 8w_1 - 4x_2 - w_3$

ratios
1 biggest, x_1 leaves
-4/96
-8/9992

$\xi = 10000 - 4x_1 - 2x_2 - w_3$
$w_1 = 1 - x_1$
$w_2 = 100 - 4x_1 - x_2$
$x_3 = 10000 - 8x_1 - 4x_2 - w_3$

optimal, with value 10000
$\vec{x} = (x_1, x_2, x_3) = (0, 0, 10000)$

Obs: If x_3 had entered in the first iteration, we would have

$w_3 = 10000 - 8x_1 - 4x_2 - x_3$

$\Downarrow$

$x_3 = 10000 - 8x_1 - 4x_2 - w_3$

$\xi = 4x_1 + 2x_2 + x_3 = 4x_1 + 2x_2 + 10000 - 8x_1 - 4x_2 - w_3$
$= 10000 - 4x_1 - 2x_2 - w_3$, optimal!

Ex 4.6

$$\xi = -\sum_{j=1}^{n} \varepsilon_j 10^{n-j}\left(\tfrac{1}{2} b_j - x_j\right)$$

$$w_i = \sum_{j=1}^{i-1} \varepsilon_i \varepsilon_j 10^{i-j}(b_j - 2x_j) + \left(b_i - x_i\right)$$

$b_i - x_i = \frac{1}{2}b_i + \frac{1}{2}b_i - x_i$

Make the substition $y_i = (x_i - \frac{1}{2} b_i)\,\varepsilon_i 10^{-i}$

$z_i = (w_i - \frac{1}{2} b_i)\,\varepsilon_i 10^{-i}$

drop 10^n

$$\xi = 10^n \sum_{j=1}^{n} y_j$$

$$z_i = -2\sum_{j=1}^{i-1} y_j - y_i$$

dictionary:

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \\ -2 & -1 & & \\ \vdots & \vdots & & \\ -2 & \cdots & -2 & -1 \\ \vdots & & & \end{bmatrix}$$

Assume y_k enters and z_k leaves:

$$z_k = -2\sum_{j=1}^{k-1} y_j - y_k \iff y_k = -2\sum_{j=1}^{k-1} y_j - z_k$$

$$\Updownarrow$$

$$-y_k = -2\sum_{j=1}^{k-1}(-y_j) - (-z_k)$$

objective:

$$\sum_{j=1}^{n} y_j = \sum_{j=1}^{k-1} y_j - 2\sum_{j=1}^{k-1} y_j - z_k + \sum_{j=k+1}^{n} y_j$$

$$= -\sum_{j=1}^{k-1} y_j - z_k + \sum_{j=k+1}^{n} y_j$$

$$= \sum_{j=1}^{k-1}(-y_j) - z_k + \sum_{j=k+1}^{n} y_j$$

constraints $1, \ldots, k-1$ are unchanged (no y_k involved):

$$z_i = -2\sum_{j=1}^{i-1} y_j \iff -z_i = -2\sum_{j=1}^{i-1}(-y_j).$$

constraints $(k+1) \cdots n$

$$z_r = -2 \sum_{j=1}^{r-1} y_j - y_r$$

$$= -2\left(\sum_{j=1}^{k-1} y_j + \left(-2 \sum_{j=1}^{k-1} y_j - z_k \right) + \sum_{j=k+1}^{n} y_j \right) - y_r$$

$$= 2 \sum_{j=1}^{k-1} y_j + 2z_k - 2 \sum_{j=k+1}^{n} y_j - y_r$$

$$= -2 \sum_{j=1}^{k-1} (-y_j) - 2(-z_k) - 2 \sum_{j=k+1}^{n} y_j - y_r$$

We see that the new system has the same coefficients as the old one, with z_j, y_j replaced with $-z_j, -y_j$ for $1 \le j \le k$

(corresponds to ϵ_j, $1 \le j \le k$ reversed)