Lecture 25/2

Duality theory part II

Question: if $\vec{x}$ is feasible for (P), and $\vec{y}$ for (D), what is required of them in order to be optimal for (P) and (D)?

Theorem 5.3 (complementary slack)

Assume $\vec{x}$ feasible for (P), $\vec{y}$ for (D), primal slack variables $w_1, \dots, w_m$, dual slack variables $z_1, \dots, z_n$. Then

$$\left.\begin{array}{l}\vec{x} \text{ optimal for (P)} \\ \vec{y} \text{ optimal for (D)}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{ll} x_j z_j = 0 & \text{for } j = 1, \dots, n \\ y_i \cdot w_i = 0 & \text{for } i = 1, \dots, m \end{array}\right.$$

complementary slack.

Proof: Proof of weak duality

$$\sum_{j=1}^{n} c_j x_j \overset{(1)}{\leq} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij}\right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i \overset{(2)}{\leq} \sum_{i=1}^{m} b_i y_i$$

$\vec{x}/\vec{y}$ optimal (for (P)/(D))

$\Updownarrow$

(1) and (2) are equalities.

(1): $\forall j: x_j = 0$ or $\underbrace{c_j = \sum_{i=1}^{m} y_i a_{ij}}_{z_j = 0} \Leftrightarrow x_j z_j = 0$

(2) $\forall j: y_i = 0$ or $\underbrace{\sum_{j=1}^{n} a_{ij} x_j = b_i}_{w_i = 0} \Leftrightarrow y_i w_i = 0.$

■

Types of algorithms:

- primal simplex: primal feasibility + complementary slack at at all steps.
- dual simplex: dual feasibility + complementary slack at all step.
- primal-dual: primal/dual feasibility at all steps, at end: complementary slack.

An LP problem can be solved by securing:

1. Primal feasibity
2. Dual feasibility
3. Complementary slack.

Example: Dual simplex method:

$$\eta = 12 - 4x_1 - x_2 - x_3$$
$$w_1 = -4 + 3x_1 - 11x_2 + x_3$$
$$w_2 = 3 - x_1 + 3x_2 - 2x_3$$

not primal feasible

dual feasible

$$-\xi = -12 + 4y_1 - 3y_2$$
$$z_1 = 4 - 3y_1 + y_2$$
$$z_2 = 1 + 11y_1 - 3y_2$$
$$z_3 = 1 - y_1 + 2y_2$$

y_1 enters

ratios

$3/4$

-11

1

1 biggest, so that z_3 leaves

"Dual pivot" can be seen as follows in the primal dictionary:

$$\eta = 12 - 4x_1 - x_2 - x_3$$
$$w_1 = -4 + 3x_1 - 11x_2 + x_3$$
$$w_2 = 3 - x_1 + 3x_2 - 2x_3$$

y_1 enters $\Rightarrow$ w_1 leaves

z_3 leaves $\Rightarrow$ x_3 enters

ratios: $\frac{3}{4}, -11, 1$

$$x_3 = 4 - 3x_1 + 11x_2 + w_1$$
$$\eta = 8 - x_1 - 12x_2 - w_1$$
$$x_3 = 4 - 3x_1 + 11x_2 + w_1$$
$$w_2 = -5 + 5x_1 - 19x_2 - 2w_1$$

w_2 leaves

ratios: $5, -\frac{19}{12}, -2$

biggest, x_1 enters

$$\eta = 7 - \frac{1}{5}w_2 - 15.8x_2 - 1.4w_1$$
$$x_3 = 1 + \cdots$$
$$x_1 = 1 + \cdots$$

This is also primal feasible, so optimal

$x_2 = 0$, $x_1 = x_3 = 1 \Rightarrow \vec{x} = (x_1, x_2, x_3) = (1, 0, 1)$

Value: 7.

Development of objective value: 12 → 8 → 7.

General procedure:

1. Replace objective so that problem is dual ~~feasible~~ → $(-x_1, -x_2 - \dots)$
2. Apply dual simplex method
3. Replace with original objective function.
4. Solve with primal simplex method.

Other LP forms

$$\max \vec{c}^T \vec{x}$$
$$\text{s.t.} \quad A\vec{x} = \vec{b}$$
$$\vec{x} \geq \vec{0}$$

↕ rewrite

$$\max \vec{c}^T \vec{x}$$
$$\text{s.t.} \quad A\vec{x} \leq \vec{b}$$
$$-A\vec{x} \leq -\vec{b}$$
$$\vec{x} \geq \vec{0}$$

↕

$$\max \vec{c}^T \vec{x}$$
$$\text{s.t} \quad \begin{bmatrix} A \\ -A \end{bmatrix} \vec{x} \leq \begin{bmatrix} \vec{b} \\ -\vec{b} \end{bmatrix}$$
$$\vec{x} \geq \vec{0}$$

↑

Dual problem

$$\min \begin{bmatrix} \vec{b} \\ -\vec{b} \end{bmatrix}^T \vec{y}$$
$$\text{s.t.} \quad \begin{bmatrix} A \\ -A \end{bmatrix}^T \vec{y} \geq \vec{c}$$
$$\vec{y} \geq \vec{0}$$

$$y = (y^+, y^-)$$

$$\min \left(\vec{b}^T y^+ - \vec{b}^T y^-\right)$$
$$\text{s.t.} \quad A^T y^+ - A^T y^- \geq \vec{c}$$
$$y^+, y^- \geq \vec{0}$$

$$\min \vec{b}^T (y^+ - y^-)$$
$$\text{s.t.} \quad A^T (y^+ - y^-) \geq \vec{c}$$
$$y^+, y^- \geq \vec{0}$$

$$y = y^+ - y^-$$

$$\min \vec{b}^T \vec{y}$$
$$\text{s.t.} \quad A^T \vec{y} \geq \vec{c}$$

Conclusion: Equality constraints in (P) $\Rightarrow$ unconstrained variables in (D)

can show also: unconstrained variables in (P) $\Rightarrow$ equality constraints in (D).

Interpretation in terms of Lagrange duality

The Lagrangian function of (P) is defined as

$$\Pi(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j + \sum_{i=1}^{m} b_i y_i$$
$$= c^T \vec{x} - \vec{y}^T A \vec{x} + \vec{b}^T \vec{y}.$$

assume $\vec{x}, \vec{y} \geq \vec{0}$.

Rewrite in two ways:

$$\Pi(x_1,\dots,x_n,y_1,\dots,y_m) = \begin{cases} \sum_{j=1}^{n}\left(c_j - \sum_{i=1}^{m} y_i a_{ij}\right)x_j + \sum_{i=1}^{m} b_i y_i \\ \sum_{i=1}^{m}\left(b_i - \sum_{j=1}^{n} a_{ij}x_j\right)y_i + \sum_{j=1}^{n} c_j x_j \end{cases}$$

$\vec{x}$ fixed: $\min_{\vec{y}\geq\vec{0}} \Pi(\vec{x},\vec{y}) = \begin{cases} -\infty & \text{if } b_i - \sum a_{ij}x_j < 0 \text{ for some } i \\ \sum_{j=1}^{n} c_j x_j & \text{if } b_i - \sum a_{ij}x_j \geq 0 \text{ for all } i \end{cases}$

primal feasibility

$\vec{y}$ fixed: $\max_{\vec{x}\geq\vec{0}} \Pi(\vec{x},\vec{y}) = \begin{cases} \infty & \text{if } c_j - \sum y_i a_{ij} > 0 \text{ for some } j \\ \sum_{i=1}^{m} b_i y_i & \text{if } c_j - \sum y_i a_{ij} \leq 0 \text{ for all } j \end{cases}$

dual feasibility

In summary:

$$\max_{x\geq 0}\min_{y\geq 0} \Pi(\vec{x},\vec{y}) = \max \sum_{j=1}^{n} c_j x_j \text{, with } \vec{x} \text{ feasible.}$$

$$\min_{y\geq 0}\max_{x\geq 0} \Pi(\vec{x},\vec{y}) = \min \sum_{i=1}^{m} b_i y_i \text{, with } \vec{y} \text{ dual feasible.}$$

This is a "saddle point interpretation" of the primal/dual problem.