Lecture 3/3

Exercise 5.1

$$
\begin{aligned}
\max \quad & x_1 - x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 \geq 0 \quad \leftarrow \text{wrong way} \\
& 4x_1 + 3x_2 + 4x_3 - 2x_4 \leq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 1 \quad \leftarrow \text{equality} \\
& x_2, x_3 \geq 0 \quad \leftarrow x_1, x_4 \text{ unconstrained}
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & x_1 - 2x_2 \\
\text{s.t.} \quad & -x_1 - 2x_2 + x_3 - x_4 \leq 0 \\
& 4x_1 + 3x_2 - 4x_3 - 2x_4 \leq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 1 \\
& x_2, x_3 \geq 0
\end{aligned}
$$

$\vec{c} = (1, -2, 0, 0)$

$A = \begin{pmatrix} -1 & -2 & 1 & -1 \\ 4 & 3 & -4 & -2 \\ -1 & -1 & 2 & 1 \end{pmatrix}$

$\vec{b} = (0, 3, 1)$

Dual problem:

$$
\begin{aligned}
\min \quad & 3y_2 + y_3 \\
\text{s.t.} \quad & -y_1 + 4y_2 - y_3 = 1 \\
& -2y_1 + 3y_2 - y_3 \geq -2 \\
& y_1 - 4y_2 + 2y_3 \geq 0 \\
& -y_1 - 2y_2 + y_3 = 0 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

x_1, x_4 unconstrained $\Leftrightarrow$ equality in constraint 1, 4 in dual.

equality in constraint 3 $\Leftrightarrow$ y_3 unconstrained in dual

Exercise 5.4

(P)

$$
\begin{aligned}
\max \quad & 2x_1 + x_2 \\
& \vdots
\end{aligned}
$$

(D)

$$
\begin{aligned}
\min \quad & 4y_1 + 3y_2 + 5y_3 + y_4 \\
& \vdots
\end{aligned}
$$

$$
\begin{aligned}
\eta &= 2x_1 + x_2 \\
w_1 &= 4 - 2x_1 - x_2 \\
w_2 &= 3 - 2x_1 - 3x_2 \\
w_3 &= 5 - 4x_1 - x_2 \\
w_4 &= 1 - x_1 - 5x_2
\end{aligned}
$$

ratios

1/2

2/3

4/5

1, biggest, w_4 leaves

$$
\begin{aligned}
-\xi &= -4y_1 - 3y_2 - 5y_3 - y_4 \\
z_1 &= -2 + 2y_1 + 2y_2 + 4y_3 + y_4 \\
z_2 &= -1 + y_1 + 3y_2 + y_3 + 5y_4
\end{aligned}
$$

$x_1 = 1 - w_4 - 5x_2$

$\eta = 2 - 2w_4 - 9x_2$
$w_1 = 2 + \dots$
$w_2 = 1 + \dots$
$w_3 = 1 + \dots$
$x_1 = 1 + \dots$

$-\xi = -2 - 2y_1 - y_2 - y_3 - z_1$
$y_4 = 2 + \dots$
$z_2 = 9 + \dots$

optimal, value 2, $\vec{x} = (x_1, x_2) = (1, 0)$ $\vec{y} = (y_1, y_2, y_3, y_4) = (0, 0, 0, 2)$

$c_1 x_1 + c_2 x_2 = 2x_1 + x_2$
$= 2 \cdot 1 + 0 = 2$

$b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_4$
$= 0 + 0 + 0 + 1 \cdot 2 = 2$

Exercise 5.3

$$\begin{aligned} \max\ & 2x_1 + 8x_2 - x_3 - 2x_4 \\ & 2x_1 + 3x_2 + 6x_4 \le 6 \\ & -2x_1 + 4x_2 + 3x_3 \le 1.5 \\ & 3x_1 + 2x_2 - 2x_3 - 4x_4 \le 4 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

a)

$$\begin{aligned} \min\ & 6y_1 + 1.5y_2 + 4y_3 \\ \text{s.t}\ & 2y_1 - 2y_2 + 3y_3 \ge 2 \\ & 3y_1 + 4y_2 + 2y_3 \ge 8 \\ & 3y_2 - 2y_3 \ge -1 \\ & 6y_1 - 4y_3 \ge -2 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

We arrive at

$\xi = 3.5 - 0.25w_1 + 6.25x_2 - 0.5w_3 - 1.5x_4$ ratio
$x_1 = 3.0 - 0.5w_1 - 1.5x_2 - 3.0x_4$ 1/2
$w_2 = 0.0 + 1.25w_1 - 3.25x_2 - 1.5w_3 + 13.5x_4$ ∞
$x_3 = 2.5 - 0.75w_1 - 1.25x_2 + 0.5w_3 - 6.5x_4$ 1/2

w_2 leaves.

b) Basic variables: x_1, w_2, x_3

Nonbasic variables: w_1, x_2, w_3, x_4.

c) Basic solution: $w_1 = x_2 = w_3 = x_4 = 0$, $x_1 = 3$, $w_2 = 0$, $x_3 = 2.5$

$$\vec{x} = (x_1, x_2, x_3, x_4) = (3, 0, 2.5, 0)$$

Dictionary is feasible, and degenerate (0 in constraint 2)

d) Dual dictionary:

$$\begin{aligned} -\eta &= -3.5 - 3.0z_1 + 0y_2 - 2.5z_3 \\ y_1 &= 0.25 + 0.5z_1 - 1.25y_2 + 0.75z_3 \\ z_2 &= -6.25 + 1.5z_1 + 3.25y_2 + 1.25z_3 \\ y_3 &= 0.5 \quad\quad + 1.5y_2 - 0.5z_3 \\ z_4 &= 1.5 + 3z_1 - 13.5y_2 + 6.5z_3 \end{aligned}$$

$x_j \leftrightarrow z_j$

$w_i \leftrightarrow y_i$

e) Basic dual solution: $y_1 = 0.25$, $y_2 = 0$, $y_3 = 0.5$, $y_4 =$

$$\Rightarrow \vec{y} = (y_1, y_2, y_3) = (0.25, 0, 0.5)$$

$$\Rightarrow \vec{z} = (z_1, z_2, z_3, z_4) = (0, -6.25, 0, 1.5)$$

$\vec{y}$ not feasible for dual problem.

f) componentwise

$$\vec{w} \cdot \vec{y} = (0,0,0) \cdot (0.25, 0, 0.5) = (0,0,0)$$

$$\vec{x} \cdot \vec{z} = (3, 0, 2.5, 0) \cdot (0, -6.25, 0, 1.5) = (0,0,0,0)$$

g) current primal solution not optimal due to 6.25 in objective.

h) x_2 will enter, w_2 leave.

We can't increase $x_2 \Rightarrow$ pivot degenerate.

Exercise 5.7

not optimal / not dual feasible

$$\begin{aligned} \max \quad & 2x_1 - 6x_2 \\ \text{s.t.} \quad & -x_1 - x_2 - x_3 \leq -2 \\ & 2x_1 - x_2 + x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

not primal feasible

replace objective with $-x_1 \; -x_2 \; -x_3$

$$\begin{aligned} \xi &= -x_1 - x_2 - x_3 \\ w_1 &= -2 + x_1 + x_2 + x_3 \\ w_2 &= 1 - 2x_1 + x_2 - x_3 \end{aligned}$$

Dual dictionary:

$$\begin{aligned} -\eta &= 2y_1 - y_2 \\ z_1 &= 1 - y_1 + 2y_2 \\ z_2 &= 1 - y_1 - y_2 \\ z_3 &= 1 - y_1 + y_2 \end{aligned}$$

feasible, y_1 enters

ratios

1

1

1

Blands rule : z_1 leaves

$$\begin{aligned} -\eta &= 2 - 2z_1 + 3y_2 \\ y_1 &= 1 - z_1 + 2y_2 \\ z_2 &= z_1 - 3y_2 \\ z_3 &= z_1 - y_2 \end{aligned}$$

y_2 enters

ratios

-2

∞

∞

Blands rule: z_2 leaves

$$\begin{aligned} -\eta &= 2 - z_1 - z_2 \\ y_1 &= 1 - \tfrac{1}{3}z_1 - \tfrac{2}{3}z_2 \\ y_2 &= \tfrac{1}{3}z_1 - \tfrac{1}{3}z_2 \\ z_3 &= \tfrac{2}{3}z_1 + \tfrac{1}{3}z_2 \end{aligned}$$

optimal!

Taking negative transpose gets us back to primal dictionary

$$\begin{aligned} \xi &= -2 - w_1 \\ x_1 &= 1 + \tfrac{1}{3}w_1 - \tfrac{1}{3}w_2 - \tfrac{2}{3}x_3 \\ x_2 &= 1 + \tfrac{2}{3}w_1 + \tfrac{1}{3}w_2 - \tfrac{1}{3}x_3 \end{aligned}$$

original objective:

$$\begin{aligned} & 2x_1 - 6x_2 \\ &= 2\left(1 + \tfrac{1}{3}w_1 - \tfrac{1}{3}w_2 - \tfrac{2}{3}x_3\right) \\ &\quad -6\left(1 + \tfrac{2}{3}w_1 + \tfrac{1}{3}w_2 - \tfrac{1}{3}x_3\right) \\ &= -4 - \tfrac{10}{3}w_1 - \tfrac{8}{3}w_2 + \tfrac{2}{3}x_3 \end{aligned}$$

primal dictionary:

$$\begin{aligned} \xi &= -4 - \tfrac{10}{3} w_1 - \tfrac{8}{3} w_2 + \tfrac{2}{3} x_3 \\ x_1 &= 1 + \tfrac{1}{3} w_1 - \tfrac{1}{3} w_2 - \tfrac{2}{3} x_3 \\ x_2 &= 1 + \tfrac{2}{3} w_1 + \tfrac{1}{3} w_2 - \tfrac{1}{3} x_3 \end{aligned}$$

ratios: $2/3$, $1/3$

$\frac{2}{3}$ biggest, x_1 leaves

$$\begin{aligned} \xi &= -3 - 3w_1 - 3w_2 - x_1 \\ x_3 &= 3/2 + \cdots \\ x_2 &= 1/2 + \cdots \end{aligned}$$

optimal!
value -3
$\vec{x} = (x_1, x_2, x_3)$
$= (0, \frac{1}{2}, \frac{3}{2})$

dual primal two phase method was applied here.