Chap.6 Simplex method in matrix notation

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\sum a_{ij} x_j \le b_i \quad i = 1, \dots, m$$

$$x_j \ge 0 \quad j = 1, \dots, n$$

define slacks $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$ $\left(x_{n+i} + \sum_{j=1}^{n} a_{ij} x_j = b_i\right)$

$x_{n+i} = w_i$

$$[A \quad I_m] \begin{bmatrix} x_1 \\ \vdots \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$ (overall i)

problem can be rewritten as

$$\max \vec{c}^{\,T} \vec{x}$$

$$\text{s.t.} \quad A\vec{x} = \vec{b}$$

$$\vec{x} \ge \vec{0}$$

new A, new $\vec{c}$

where $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+m} \end{pmatrix}$

Let B be the current basis in simplex, and set $\vec{x} = \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix}$ (basic variables first), $A = [B \quad N]$, where we have interchanged the columns of A so that columns corresponding to basic variables come first. We have

$$A\vec{x} = [B \quad N] \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix} = B\vec{x}_B + N\vec{x}_N = \vec{b}$$

if B is invertible: $\vec{x}_B = B^{-1}\vec{b} - B^{-1}N\vec{x}_N$

Set also $\vec{c} = \begin{bmatrix} \vec{c}_B \\ \vec{c}_N \end{bmatrix}$, so that $\vec{c}^{\,T}\vec{x} = \begin{bmatrix} \vec{c}_B \\ \vec{c}_N \end{bmatrix}^T \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix}$

$$= \vec{c}_B^{\,T}\vec{x}_B + \vec{c}_N^{\,T}\vec{x}_N$$

$$= \vec{c}_B^{\,T}(B^{-1}\vec{b} - B^{-1}N\vec{x}_N) + \vec{c}_N^{\,T}\vec{x}_N$$

$$= \vec{C}_B^T B^{-1}\vec{b} - (\vec{C}_B^T B^{-1} N - \vec{C}_N)\vec{X}_N$$

In summary:

(*P)

$$\xi = \vec{C}_B^T B^{-1}\vec{b} - (\vec{C}_B^T B^{-1} N - \vec{C}_N)\vec{X}_N$$
$$\vec{X}_B = B^{-1}\vec{b} - B^{-1} N \vec{X}_N$$

new notation

$$\xi = \bar{\xi} + \sum_{j=1}^{n} \bar{c}_j x_j$$
$$\vec{X}_B = \bar{\vec{b}} - \bar{A}\vec{X}_N$$

old notation

Compare new and old notation:

$$\vec{C}_B^T B^{-1}\vec{b} = \bar{\xi}$$
$$\vec{C}_N - \vec{C}_B^T B^{-1} N = \bar{c}$$
$$B^{-1}\vec{b} = \bar{\vec{b}}$$
$$-B^{-1} N = -\bar{A}$$

basic solution

$$\vec{X}_B^* = B^{-1}\vec{b}$$
$$\vec{X}_N^* = \vec{0}$$

Matrix form of the dual problem

Since dual variables are complementary to primal variables, we rename variables as follows:

$$(z_1, \dots, z_n, y_1, \dots, y_m) \to (z_1, \dots, z_n, z_{n+1}, \dots, z_{n+m})$$

which are complementary to $(x_1, \dots, x_n, w_1, \dots, w_m) \to (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$

Dual dictionary: Take negative transpose of (*P)

(*D)

$$-\eta = -\vec{C}_B B^{-1}\vec{b} - (B^{-1}\vec{b})^T \vec{Z}_B$$
$$\vec{Z}_N = (B^{-1}N)^T \vec{C}_B - \vec{C}_N + (B^{-1}N)^T \vec{Z}_B$$

$$\vec{Z}_B^* = \vec{0}$$
$$\vec{Z}_N^* = (B^{-1}N)^T \vec{C}_B - \vec{C}_N$$

To simplify (*P) and (*D): set $\xi^* = \vec{C}_B^T B^{-1}\vec{b}$, and use expressions above:

$$\xi = \xi^* - (\vec{Z}_N^*)^T \vec{X}_N$$
$$\vec{X}_B = \vec{X}_B^* - (B^{-1}N)^T \vec{X}_N$$

$$-\eta = -\xi^* - (\vec{X}_B^*)^T \vec{Z}_B$$
$$\vec{Z}_N = \vec{Z}_N^* + (B^{-1}N)^T \vec{Z}_B$$

Summary:

- Partition the $n+m$ variables into m basic and n nonbasic variables
- Compute x_B^*, z_N^* ($\vec{x}_N = \vec{z}_B = \vec{0}$) using (*P) and (*D)
- Assume $x_B^* \geq \vec{0}$ (primal feasibility.

Steps:

1. If $z_N^* \geq 0$ stop (dual feasibility $\Rightarrow$ optimality)
2. Select entering variable x_j so that $z_j^* < 0$ (x_j enters in P)
3. Compute primal step direction $\Delta \vec{x}_B$:

   Increase x_j, keep others at 0, i.e. $\vec{x}_N = t\vec{e}_j$, $t \geq 0$:

   $$\vec{x}_B = \vec{x}_B^* - t(B^{-1}N)^T \vec{e}_j$$

   $\Delta \vec{x}_N$
4. Compute primal step length: Largest so that $\vec{x}_B \geq \vec{0}$:

   $$\vec{x}_B^* - t\Delta \vec{x}_B \geq \vec{0}$$

   $$\vec{x}_B^* \geq t\Delta \vec{x}_B$$

   $$\frac{1}{t} \geq \frac{(\Delta \vec{x}_B)_i}{x_i^*} \quad \text{for all } i.$$

   i.e. choose t so that $\frac{1}{t} = \max_{i \in B} \frac{(\Delta \vec{x}_B)_i}{x_i^*} \Rightarrow t = \frac{1}{\max \frac{(\Delta \vec{x}_B)_i}{x_i^*}}$
5. Select leaving variable: Any $i \in B$ that achieves max in 4. (i enters in (D)).
6. Compute (dual step direction): $\Delta \vec{z}_N = -(B^{-1}N)^T \vec{e}_i$
7. Compute dual step length: if j leaves in (D):

   $$0 = z_j = z_j^* - s\Delta \vec{z}_j \Rightarrow s = \frac{z_j^*}{\Delta \vec{z}_j}$$
8. Update current primal and dual solutions:

   $x_j^* = t$ $\qquad$ $z_i^* = s$

$$x_B^* := x_B^* - t\,\Delta\vec{x}_B \qquad z_N^* := z_N^* - s\cdot\Delta\vec{z}_N$$

a. Update basis: In B, replace i with j
In N, replace j with i.

Section 7.1: Sensitivity analysis

Question: How much can we change $A, \vec{b}, \vec{c}$ in an LP problem without affecting the optimality of a basis?

Assume B optimal basis for (P). From chap. 6:

$$\vec{x}_B^* = B^{-1}\vec{b}$$

$$\vec{z}_N^* = (B^{-1}N)^T\vec{c}_B - \vec{c}_N$$

$$\xi^* = \vec{c}_B^{\,T} B^{-1}\vec{b}$$

Changes in $\vec{c}$ affects only $\vec{z}_N^*$ and ξ^*
it follows: dictionary primal feasible
new dictionary $\Downarrow$ primal feasible.

Changes in $\vec{b}$ affect only $\vec{x}_B^*$ and ξ^*
it follows:
dictionary dual feasible
$\Downarrow$
new dictionary dual feasible

Consider a change in $\vec{c}$: $\vec{c} \to \vec{c} + t\Delta\vec{c}$

$$\vec{z}_N^* = (B^{-1}N)^T(\vec{c}_B + t\Delta\vec{c}_B) - (\vec{c}_N + t\Delta c_N)$$

$$= \underbrace{(B^{-1}N)^T\vec{c}_B - \vec{c}_N}_{z_N^*} + t\,\underbrace{((B^{-1}N)^T\Delta\vec{c}_B - \Delta\vec{c}_N)}_{\text{Defined as } \Delta\vec{z}_N}$$

$$= z_N^* + t\,\Delta\vec{z}_N$$

So, dictionary remains dual feasible if (*) $\boxed{z_N^* + t\,\Delta\vec{z}_N \geq 0}$

Sensitivity analysis: Find an interval for t so that (*) holds.

Example section 7.1

$$
\begin{aligned}
\max \quad & 5x_1 + 4x_2 + 3x_3 \\
\text{s.t} \quad & 2x_1 + 3x_2 + x_3 \le 5 \\
& 4x_1 + x_2 + 2x_3 \le 11 \\
& 3x_1 + 4x_2 + 2x_3 \le 8
\end{aligned}
$$

$w_1 = x_4$, $w_2 = x_5$, $w_3 = x_6$

optimal dictionary:

$$
\begin{aligned}
\zeta &= 13 - 3x_2 - x_4 - x_6 \\
x_3 &= 1 + x_2 + 3x_4 - 2x_6 \\
x_1 &= 2 - 2x_2 - 2x_4 + x_6 \\
x_5 &= 1 + 5x_2 + 2x_4
\end{aligned}
$$

$B\vec{x}_B + N\vec{x}_N = \vec{b}$

$\vec{x}_B = B^{-1}\vec{b} - B^{-1}N\vec{x}_N$

$\Rightarrow -B^{-1}N = \begin{pmatrix} 1 & 3 & -2 \\ -2 & -2 & 1 \\ 5 & 2 & 0 \end{pmatrix}$

$B = \{3,1,5\}$, $N = \{2,4,6\}$, $z_N^* = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

How much can we change c_1 without affecting the optimality of B?

$\vec{c} = (5,4,3,0,0,0)$ $\quad \Delta\vec{c} = (1,0,0,0,0,0)$

$\Delta\vec{c}_B = (0,1,0)$

$\Delta\vec{c}_N = (0,0,0)$

$$
\Delta\vec{z}_N = (B^{-1}N)^T \Delta\vec{c}_B - \Delta\vec{c}_N = \begin{pmatrix} -1 & 2 & -5 \\ -3 & 2 & -2 \\ 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}
$$

$-t\Delta z_N \le z_N^*$:

$$
\begin{cases} -2t \le 3 \\ -2t \le 1 \\ t \le 1 \end{cases} \iff \begin{cases} t \ge 3/2 \\ t \ge -\frac{1}{2} \\ t \le 1 \end{cases} \iff t \in [-\tfrac{1}{2}, 1]
$$