Exercise 5.12

$$\max_{x \geq 0} \min_{y \geq 0} (x - y) \qquad \min_{y \geq 0} \max_{x \geq 0} (x - y)$$

$$\min_{y \geq 0} (x - y) = -\infty \qquad \max_{x \geq 0} (x - y) = \infty$$

$$= \max_{x \geq 0} (-\infty) = -\infty \qquad = \min_{y \geq 0} \infty = \infty$$

So, primal/dual problems have different optimal values!

Here the Lagrangian function $A = [0]$, $\vec{c} = [1]$, $\vec{b} = [-1]$

$$\Pi(x_1, \dots, x_n, y_1, \dots, y_m) = \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} \sum_{i=1}^{m} y_i a_{ij} x_j + \sum_{i=1}^{m} b_i y_i$$

primal problem

$$\begin{aligned} &\max \quad x \\ &\text{s.t.} \quad 0 \leq -1 \end{aligned}$$

infeasible

dual problem

$$\begin{aligned} &\min \quad -y \\ &\text{s.t.} \quad 0 \geq 1 \end{aligned}$$

infeasible

Exercise 5.15

(P)

$$\begin{aligned} &\max \quad \vec{p}^{\,T} \vec{x} \\ &\text{s.t} \quad \begin{bmatrix} \vec{q}^{\,T} \\ I_n \end{bmatrix} \vec{x} \leq \begin{bmatrix} \beta \\ \vdots \\ 1 \end{bmatrix} \\ &\qquad\qquad \vec{x} \geq \vec{0} \end{aligned}$$

$$\begin{aligned} \xi &= 0 + p_1 x_1 + \cdots + p_n x_n \\ w_0 &= \beta - q_1 x_1 - \cdots - q_n x_n \\ w_1 &= 1 - x_1 \\ &\vdots \\ w_n &= 1 - x_n \end{aligned}$$

(D)

$$\begin{aligned} &\min \quad \begin{bmatrix} \beta \\ \vdots \\ 1 \end{bmatrix}^T \vec{y} \\ &\text{s.t.} \quad \begin{bmatrix} \vec{q} & I_n \end{bmatrix} \vec{y} \geq \vec{p} \\ &\qquad\qquad \vec{y} \geq \vec{0} \end{aligned}$$

$$\begin{aligned} -\eta &= 0 - \beta y_0 - y_1 - \cdots - y_n \\ z_1 &= -p_1 + q_1 y_0 + y_1 \\ &\vdots \\ z_n &= -p_n + q_n y_0 + y_n \end{aligned}$$

Will show that $\vec{x}/\vec{y}$ (as stated in book) are feasible, and satisfy complementary slack, so they are primal/dual optimal.

Recall that $k = \min\{j : q_{j+1} + \cdots + q_n \le B\}$

$$\vec{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{B - q_{k+1} - \cdots - q_n}{q_k} \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{at place } k$$

$$w_0 = B - \sum_{j=1}^{n} q_j x_j = B - q_k\left(\frac{B - q_{k+1} - \cdots - q_n}{q_k}\right) - \sum_{j=k+1}^{n} q_j$$

$$= B - (B - q_{k+1} - \cdots - q_n) - \sum_{j=k+1}^{n} q_j$$

$$= \sum_{j=k+1}^{n} q_j - \sum_{j=k+1}^{n} q_j = 0$$

$$w_1 = \cdots w_{n-1} = 1$$

$$w_k = 1 - x_k = 1 - \frac{B - q_{k+1} - \cdots - q_n}{q_k}$$

$$= \frac{q_k - B + \sum_{j=k+1}^{n} q_j}{q_k}$$

$$= \frac{\sum_{j=k}^{n} q_j - B}{q_k} \ge 0$$

$$w_{k+1} = \cdots = w_n = 1 - 1 = 0$$

Thus, $\vec{x}$ is primal feasible.

$$\vec{w} \cdot \vec{q} = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{k-1} \\ w_k \\ w_{k+1} \\ \vdots \\ w_n \end{pmatrix} \cdot \begin{pmatrix} p_k/q_k \\ 0 \\ \vdots \\ 0 \\ q_j\left(\frac{p_j}{q_j} - \frac{p_k}{q_k}\right) \end{pmatrix} \quad j > k = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus, we have first part of complementary slack.

$$z_j = -p_j + q_j y_0 + y_j = \begin{cases} j \le k: & -p_j + q_j \frac{p_k}{q_k} + 0 = q_j\left(\frac{p_k}{q_k} - \frac{p_j}{q_j}\right) \ge 0 \\ j > k: & -p_j + q_j \frac{p_k}{q_k} + q_j\left(\frac{p_j}{q_j} - \frac{p_k}{q_k}\right) = 0 \end{cases}$$

Thus $\vec{z} \geq \vec{0}$, and clearly $\vec{y} \geq \vec{0}$, so that $\vec{y}$ is dual feasible.
Also

$$\vec{z} \cdot \vec{x} = \begin{pmatrix} z_1 \\ \vdots \\ z_{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_k \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

This proves second part of complementary slack, it follows that $\vec{x}/\vec{y}$ are primal/dual optimal.