Exercise 6.1

$$\begin{aligned} \max \quad & -6x_1 + 32x_2 - 9x_3 \\ \text{s.t.} \quad & -2x_1 + 10x_2 - 3x_3 \le -6 \\ & x_1 - 7x_2 + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

↓ after pivots

$$\begin{aligned} \xi &= 18 - 3x_4 + 2x_2 \\ x_3 &= 2 - x_4 + 4x_2 - 2x_5 \\ x_1 &= \quad 2x_4 - x_2 + 3x_5 \end{aligned}$$

x_4, x_5 slacks

$-B^{-1}N$

$$= \begin{pmatrix} -1 & 4 & -2 \\ 2 & -1 & 3 \end{pmatrix}$$

a) Basic: x_3, x_1 $\quad B = \{3, 1\}$

Nonbasic: x_4, x_2, x_5 $\quad N = \{4, 2, 5\}$

b) $x_B^* = (x_3, x_1) = (2, 0)$

c) $z_N^* = (3, -2, 0)$

d) $B\vec{x}_B + N\vec{x}_N = \vec{b} \Rightarrow \vec{x}_B = B^{-1}\vec{b} - B^{-1}N\vec{x}_N$

$$A = \begin{bmatrix} -2 & 10 & -3 & 1 & 0 \\ 1 & -7 & 2 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} \Rightarrow B^{-1}N = \begin{bmatrix} 1 & -4 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

e) $x_B^* \ge 0 \Rightarrow$ primal feasible dictionary

f) Not optimal since $z_N^* \ge 0$ is not the case

g) Degenerate dictionary (second constraint).

Exercise 6.6

$$\max \vec{c}^T \vec{x} \quad \text{s.t.} \quad \vec{a} \le A\vec{x} \le \vec{b}, \quad \vec{l} \le \vec{x} \le \vec{u}$$

$$\leftrightarrow \quad \max \vec{c}^T \vec{x} \quad \text{s.t.} \quad -A\vec{x} \le -\vec{a}, \; A\vec{x} \le \vec{b}, \; -\vec{x} \le -\vec{l}, \; \vec{x} \le \vec{u}$$

$$\leftrightarrow \quad \max \vec{c}^T \vec{x} \quad \text{s.t.} \quad \begin{bmatrix} -A \\ A \\ -I \\ I \end{bmatrix} \vec{x} \le \begin{bmatrix} -\vec{a} \\ \vec{b} \\ -\vec{l} \\ \vec{u} \end{bmatrix}$$

Dual problem: since $\vec{x}$ is unconstrained $\Rightarrow$ equality constraints in dual:

$$\min \begin{bmatrix} -\vec{a} \\ \vec{b} \\ -\vec{l} \\ \vec{u} \end{bmatrix}^T \vec{y}$$

$$\begin{bmatrix} -A^T & A^T & -I & I \end{bmatrix} \vec{y} = \vec{c}, \quad \vec{y} \ge 0$$

Exercise 7.1

Initial problem:

$$\max \; x_1 + 2x_2 + x_3 + x_4$$

$$\text{s.t.} \quad 2x_1 + x_2 + 5x_3 + x_4 \le 8$$
$$2x_1 + 2x_2 + 4x_4 \le 12$$
$$3x_1 + x_2 + 2x_3 \le 18$$
$$x_1, x_2, x_3, x_4 \ge 0$$

$$A = \begin{bmatrix} 2 & 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Final optimal dictionary:

$$\begin{aligned} \xi &= 12.4 - 1.2x_1 - 0.2x_5 - 0.9x_6 - 2.8x_4 \\ x_2 &= 6 - x_1 - 0.5x_6 - 2x_4 \\ x_3 &= 0.4 - 0.2x_1 - 0.2x_5 + 0.1x_6 + 0.2x_4 \\ x_7 &= 11.2 - 1.6x_1 + 0.4x_5 + 0.3x_6 + 1.6x_4 \end{aligned}$$

slacks: x_5, x_6, x_7

here: $B = \{2, 3, 7\}$

$N = \{1, 5, 6, 4\}$

a) Objective changes from $\vec{c} = (1,2,1,1,0,0,0)$ to $(3,2,1,1,0,0,0)$:

$\Delta\vec{c} = (2,0,0,0,0,0,0)$, $\Delta\vec{c}_B = (0,0,0)$, $\Delta\vec{c}_N = (2,0,0,0)$

coefficient matrix in dictionary: $-B^{-1}N = \begin{pmatrix} -1 & 0 & -0.5 & -2 \\ -0.2 & -0.2 & 0.1 & 0.2 \\ -1.6 & 0.4 & 0.3 & 1.6 \end{pmatrix}$

When $\vec{c}$ changes, $\vec{z}_N^*$ changes, but $\vec{x}_B^*$ does not (primal feasibility preserved)

Here $\vec{z}_N^* = (1.2, 0.2, 0.9, 2.8)$

(7.1) in book: $\Delta\vec{z}_N = (B^{-1}N)^T \underbrace{\Delta\vec{c}_B}_{0} - \Delta\vec{c}_N = (-2, 0, 0, 0)$

$\vec{z}_N = \vec{z}_N^* + \Delta\vec{z}_N = (1.2, 0.2, 0.9, 2.8) + (-2, 0, 0, 0) = (-0.8, 0.2, 0.9, 2.8)$,
which is not dual feasible.

objective value: $\vec{c}_B^{\,T} B^{-1} \vec{b}$, so it changes with $\underbrace{\Delta\vec{c}_B}_{0}^{\,T} B^{-1}\vec{b} = 0$,

so objective value does not change.

new dictionary: (only change)

$$\xi = 12.4 + 0.8x_1 - 0.2x_5 - 0.9x_6 - 2.8x_4$$
$$x_2 = 6 \quad < \quad x_1 - \cdots \qquad \text{ratios } 1/6$$
$$x_3 = 0.4 - 0.2x_1 - \cdots \qquad 1/2 \text{ biggest, } x_3 \text{ leaves}$$
$$x_7 = 11.2 - 1.6x_1 - \cdots \qquad 1.6/11.2$$

$$\xi = 14 - 4x_3 - x_5 - 0.5x_6 - 2x_4$$
$$x_2 = 4 + \cdots$$
$$x_1 = 2 + \cdots$$
$$x_7 = 8 + \cdots$$

optimal, value 14, $\vec{x} = (x_1, x_2, x_3, x_4) = (2, 4, 0, 0)$

b) Objective changes from $(1.2, 1.1, 0, 0, 0)$ to $(1, 2, 0.5, 1, 0, 0)$

B and N as in a) $B = \{2, 3, 7\}$, $N = \{1, 5, 6, 4\}$

$\Delta\vec{c} = (0, 0, -0.5, 0, 0, 0, 0)$, $\Delta\vec{c}_B = (0, -0.5, 0)$, $\Delta\vec{c}_N = (0, 0, 0, 0)$

$$\Delta\vec{z}_N = \underbrace{(B^{-1}N)}_{\text{as before}}^T \Delta\vec{c}_B - \Delta\vec{c}_N = \begin{bmatrix} 1 & 0.2 & 1.6 \\ 0 & 0.2 & -0.4 \\ 0.5 & -0.1 & -0.3 \\ 2 & -0.2 & -1.6 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1 \\ -0.1 \\ 0.05 \\ 0.1 \end{bmatrix}$$

$\vec{z}_N = \vec{z}_N^* + \Delta\vec{z}_N = (1.2, 0.2, 0.9, 2.8) + (-0.1, -0.1, 0.05, 0.1)$

$= (1.1, 0.1, 0.95, 2.9)$

This is feasible, so dictionary is still optimal.

new optimal value:

$$\xi = \xi^* + \Delta c_B^T \underbrace{B^{-1}\vec{b}}_{\vec{x}_B^*} = 12.4 + [0 \ -0.5 \ 0]\begin{bmatrix} 6 \\ 0.4 \\ 2 \end{bmatrix} = 12.4 - 0.2 = 12.2$$

same optimal solution $\vec{x}$.

(or $B = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 0 & 0.5 & 0 \\ 0.2 & -0.1 & 0 \\ -0.4 & -0.3 & 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 8 \\ 12 \\ 18 \end{pmatrix}$

$B^{-1}\vec{b} = \ldots$

If $\vec{b}$ changes from $(8, 12, 18)$ to $(8, 26, 18) \Rightarrow \Delta\vec{b} = (0, 14, 0)$

Dual feasibility preserved, z_N^* does not change.

From dictionary: $x_B^* = (6, 0.4, 11.2)$ (will change when $\vec{b}$ changes)

$\Delta\vec{x}_B = B^{-1}\Delta\vec{b}$ (since $x_B^* = B^{-1}\vec{b}$)

$$= \begin{pmatrix} 0 & 0.5 & 0 \\ 0.2 & -0.1 & 0 \\ -0.4 & -0.3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 14 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -1.4 \\ -4.2 \end{pmatrix}$$

$$\vec{x}_B = \vec{x}_B^* + \Delta\vec{x}_B = \begin{pmatrix} 6 \\ 0.4 \\ 11.2 \end{pmatrix} + \begin{pmatrix} 7 \\ -1.4 \\ -4.2 \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \\ 7 \end{pmatrix}$$ ← which is not feasible

Updated objective value: $\xi = \xi^* + \vec{c}_B^T B^{-1}\Delta\vec{b}$

$$= 12.4 + (2 \ 1 \ 0)\begin{pmatrix} 7 \\ -1.4 \\ -4.2 \end{pmatrix} = 12.4 + 14 - 1.4 = \underline{25}$$

new primal dictionary:

$$\xi = 25 - 1.2x_1 - 0.2x_5 - 0.9x_6 - 2.8x_4$$
$$x_2 = 13 + \ldots$$
$$x_3 = -1 + \ldots$$
$$x_7 = 7 + \ldots$$

not feasible, but dual feasible

I neg. transpose

$$
\begin{aligned}
-\eta &= -25 - 13z_2 + z_3 - 7z_7 \\
z_1 &= 1.2 + z_2 + 0.2z_3 + 1.6z_7 \\
z_5 &= 0.2 + 0.2z_3 - 0.4z_7 \\
z_6 &= 0.9 + 0.5z_2 - 0.1z_3 - 0.3z_7 \\
z_4 &= 2.8 + 2z_2 - 0.2z_3 - 1.6z_7
\end{aligned}
$$

ratios

$-0.2/1.2 = -1/6$

-1

$1/9$ biggest, z_6 leave

$1/14$

$$
\begin{aligned}
-\eta &= -16 - 8z_2 - 10z_6 - 10z_7 \\
z_1 &= 3 + \dots \\
z_5 &= 2 + \dots \\
z_3 &= 9 + \dots \\
z_4 &= 1 + \dots
\end{aligned}
$$

optimal !

$$
\begin{aligned}
\xi &= 16 - 3x_1 - 2x_5 - 9x_3 - x_4 \\
x_2 &= 8 + \dots \\
x_6 &= 10 + \dots \\
x_7 &= 10 + \dots
\end{aligned}
$$

$\xi = 16$ optimal value

$\vec{x} = (x_1, x_2, x_3, x_4)$

$= (0, 8, 0, 0)$