Chap. 11, Game theory

The paper-scissors-rock game:

- Two players choose independently one of paper / scissors / rock
  (1 / 2 / 3)
- Rule 1: paper beats rock
  2. Rock beats scissors
  3. Scissors beats paper.

Choice of first / second player: Interpret as a row / column in a matrix
(R: row player; K: column player)

The matrix in question is called a payoff matrix, A.

entry (i,j) in A (a_{ij}): What R pays to K when R chooses i, K chooses j.

$a_{ij} < 0$: payment goes the other way.

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

R chooses paper
K chooses scissors
K wins
R pays 1 to K

R chooses paper
K chooses rock
R wins
K pays 1 to R

Can be generalised: $\binom{R}{K}$ choose among $\binom{m}{n}$ objects ($\Rightarrow A$ is $m \times n$)

Matrix games

pure / deterministic strategy: R / K play according to a fixed choice.
No such strategy guaranteed to win.

Randomized strategy:
- R makes his choice according to a given probability distribution:
  y_i: probability that R chooses i: $y_i \geq 0$, $\sum_{i=1}^{m} y_i = 1$
  $\vec{y} = (y_1, \ldots, y_m)$ is called a stochastic vector

- K makes similar choice, with probability distribution $\vec{x}$.

Expected payoff from R to K : $\sum_{i,j} y_i a_{ij} x_j = \vec{y}^T A \vec{x}$.

Question Assume K knows the strateg $\vec{y}$ of R. How should K choose his own strategy, in order to maximise payments to himself, i.e. maximise expected payoff?

$$\begin{aligned} &\max_{\vec{x} \geq 0} \ \vec{y}^T A \vec{x} \\ &\vec{e}^T \vec{x} = 1 \end{aligned}$$

If R knows K chooses his strategy like this, how should he choose $\vec{y}$ inorder to minimise his own payments?

(*)
$$\begin{aligned} &\min_{\vec{y} \geq \vec{0}} \ \max_{\vec{x} \geq \vec{0}} \ \vec{y}^T A \vec{x} \\ &\vec{e}^T \vec{y} = 1 \quad \vec{e}^T \vec{x} = 1. \end{aligned}$$

Assume instead R knows strategy $\vec{x}$ of K. How should R choose his own strategy inorder to minimise his own payment?

$$\begin{aligned} &\min_{y \geq 0} \ y^T A x \\ &e^T y = 1 \end{aligned}$$

If K knows R chooses his strategy like this, how should he choose $\vec{x}$ in order to maximise payments to himself?

(**)
$$\begin{aligned} &\max_{x \geq 0} \ \min_{y \geq 0} \ y^T A x \\ &e^T x = 1 \quad e^T y = 1 \end{aligned}$$

Rewriting (*) and (**) as LP problems:

Inner part of (*): $\max_{x \geq 0,\ e^T x = 1} y^T A x$ — clearly optimum exists, and is attained for some $\vec{e}_i$

Similarly for the inner part of (**), so

(*) $\min_{y \geq 0,\ e^T y = 1} \left(\max_i y^T A e_i \right)$ (**) $\max_{x \geq 0,\ e^T x = 1} \left(\min_i e_i^T A x \right)$

Rewrite:

(*)
$$\min u \quad \text{s.t. } u \geq y^T A e_i \ \forall i, \quad e^T y = 1, \quad y \geq 0$$

(**)
$$\max v \quad \text{s.t. } v \leq e_i^T A x \ \forall i, \quad e^T x = 1, \quad x \geq 0$$

$-e_i^T A^T y + u \geq 0$

$-A^T y + \begin{pmatrix} u \\ \vdots \\ u \end{pmatrix} \geq 0$

(*)
$$\min \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} y \\ u \end{bmatrix} \quad \text{s.t. } \begin{bmatrix} -A^T & \vec{e} \\ \vec{e}^T & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \geq \begin{bmatrix} \vec{0} \\ 1 \end{bmatrix}, \quad y \geq 0,\ u \text{ free}$$

(**)
$$\max \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ v \end{bmatrix} \quad \text{s.t. } \begin{bmatrix} -A & \vec{e} \\ \vec{e}^T & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \leq \begin{matrix} \vec{0} \\ 1 \end{matrix}, \quad x \geq 0,\ v \text{ free}$$

Dual problems.

Theorem 11.1: Minmax theorem

There exist stochastic vectors x^*, y^* so that

$$\max_{x \geq 0,\ e^T x = 1} (y^*)^T A \vec{x} = \min_{y \geq 0,\ e^T y = 1} y^T A x^*$$

Proof: Both (*) and (**) are feasible.

Duality theory: They have same optimal value $u^* = v^*$

v^* is achieved in (**) for some $\begin{pmatrix} x^* \\ v^* \end{pmatrix}$

$\Downarrow$

$$v^* = \min_i e_i^T A x^* = \min_{\substack{y \geq 0 \\ e^T y = 1}} y^T A x^*$$

u^* is achieved in (*) for some $\begin{pmatrix} y^* \\ u^* \end{pmatrix}$

$\Downarrow$

$$u^* = \max_i (y^*)^T A e_i = \max_{\substack{x \geq 0 \\ e^T x = 1}} (y^*)^T A x$$

Since $u^* = v^*$:

$$\max_{\substack{x \geq 0 \\ e^T x = 1}} (y^*)^T A x = \min_{\substack{y \geq 0 \\ e^T y = 1}} y^T A x^*$$

■

$u^* = v^*$ is called the value of the game.

If $\overset{R}{k}$ adopts $\begin{matrix} y^* \\ x^* \end{matrix}$, he pays/receives v^* per round on average.

A fair game is a game of value 0.

Interpretation: Since

$$\max_{\substack{x \geq 0 \\ e^T x = 1}} \min_{\substack{y \geq 0 \\ e^T y = 1}} y^T A x = \min_{\substack{y \geq 0 \\ e^T y = 1}} \max_{\substack{x \geq 0 \\ e^T x = 1}} y^T A x$$

Let $(x', y') \neq (x^*, y^*)$, then

$$\min_{\substack{y \geq 0 \\ e^T y = 1}} y^T A x' \leq \max_{\substack{x \geq 0 \\ e^T x = 1}} (y')^T A x$$

Thus: It is better for a player to know the other player's distribution, than the other way around, but for x^*, y^* this does not matter.

Example: paper/scissors/rock with payoff matrix $A = \begin{pmatrix} 0 & 1 & -2 \\ -3 & 0 & 4 \\ 5 & -6 & 0 \end{pmatrix}$

(**) takes form:

$$\max \quad v$$

$$\text{s.t.} \quad \left(\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 3 & 0 & -4 & 1 \\ -5 & 6 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ v \end{pmatrix} \begin{matrix} \leq \\ \leq \\ \leq \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{matrix} \vec{x} \geq \vec{0} \\ v \text{ free} \end{matrix}$$

$$
\begin{aligned}
\max\ & v \\
\text{s.t}\ & -x_2 + 2x_3 + v \le 0 \\
& 3x_1 - 4x_3 + v \le 0 \\
& -5x_1 + 6x_2 + v \le 0 \\
& x_1 + x_2 + x_3 = 1 \\
& v \text{ free}, x \ge 0
\end{aligned}
\quad \text{s.t} \quad \rightarrow \quad
\begin{aligned}
\max\ & v \\
\text{s.t.}\ & -2x_1 - 3x_2 + v \le -2 \quad x_4 \\
& 7x_1 + 4x_2 + v \le 4 \quad x_5 \\
& -5x_1 + 6x_2 + v \le 0 \quad x_6 \\
& x_1 + x_2 \le 1 \quad x_3 \\
& v \text{ free}, x_1, x_2 \ge 0
\end{aligned}
$$

Dictionary:

$$
\begin{aligned}
\xi &= v \\
x_4 &= -2 + 2x_1 + 3x_2 - v \\
x_5 &= 4 - 7x_1 - 4x_2 - v \\
x_6 &= 5x_1 - 6x_2 - v \\
x_3 &= 1 - x_1 - x_2 \\
& x \ge 0, v \text{ free}
\end{aligned}
$$

Let v enter, and see what happens.

x_4 leaves (most infeasible)

$\Rightarrow v = -2 + 2x_1 + 3x_2 - x_4$

$$
\begin{aligned}
\xi &= -2 + 2x_1 + 3x_2 - x_4 \\
v &= -2 + 2x_1 + 3x_2 - x_4 \\
x_5 &= 6 - 9x_1 - 7x_2 + x_4 \\
x_6 &= 2 + 3x_1 - 9x_2 + x_4 \\
x_3 &= 1 - x_1 - x_2 \\
& x \ge 0, v \text{ free}
\end{aligned}
$$

← can be removed, since v free.

$$
\begin{aligned}
\xi &= -2 + 2x_1 + 3x_2 - x_4 \\
x_5 &= 6 - 9x_1 - 7x_2 - x_4 \\
x_6 &= 2 + 3x_1 - 9x_2 + x_4 \\
x_3 &= 1 - x_1 - x_2 \\
& x \ge 0
\end{aligned}
$$

$$
\begin{aligned}
102\xi &= -16 - 27x_5 - 13x_6 - 62x_4 \\
102x_1 &= 40 + \dots \\
102x_2 &= 36 + \dots \\
102x_3 &= 26 + \dots \\
& x \ge 0
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
x_5 &\rightarrow z_5 = y_2 \\
x_6 &\rightarrow z_6 = y_3 \\
x_4 &\rightarrow z_4 = y_1
\end{aligned}
$$

optimal, with value $-\frac{16}{102} = u^* = v^*$

$$\vec{x} = (x_1, x_2, x_3) = \left(\frac{40}{102}, \frac{36}{102}, \frac{26}{102}\right)$$

$$\vec{y} = (y_1, y_2, y_3) = \left(\frac{62}{102}, \frac{27}{102}, \frac{13}{102}\right)$$