Exercises chap. 17

Ex. 17.1

(P)

$$\begin{aligned} \max \quad & -x_1 + x_2 \\ \text{s.t.} \quad & x_2 \le 1 \\ & -x_1 \le -1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

(D)

$$\begin{aligned} \min \quad & y_1 - y_2 \\ \text{s.t.} \quad & -y_2 \ge -1 \\ & y_1 \ge 1 \\ & y_1, y_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & -y_1 + y_2 \\ \text{s.t.} \quad & y_2 \le 1 \\ & -y_1 \le -1 \\ & y_1, y_2 \ge 0 \end{aligned}$$

Vi ser at (P) og (D) er like, slik at $\vec{x} = \vec{y}$, $\vec{w} = \vec{z}$

Løsning barrier problem

Like:
$$\begin{aligned} Ax + w &= b \\ A^T y - z &= c \\ x_j z_j &= \mu \\ y_i w_i &= \mu \end{aligned}$$

$x = y, z = w$

$$\begin{aligned} Ax + w &= b \\ x_j w_j &= \mu \end{aligned} \longrightarrow \begin{aligned} x_2 + w_1 &= 1 \\ -x_1 + w_2 &= -1 \\ x_1 w_1 = x_2 w_2 &= \mu \end{aligned}$$

$w_1 = 1 - x_2$

$$x_1 w_1 = \mu \Rightarrow x_1 (1 - x_2) = \mu$$

$w_2 = -1 + x_1$

$$x_2 w_2 = \mu \Rightarrow x_2 (-1 + x_1) = \mu \Rightarrow x_2 = \frac{\mu}{x_1 - 1}$$

$$x_1 \left(1 - \frac{\mu}{x_1 - 1}\right) = \mu \longrightarrow x_1 (x_1 - 1 - \mu) = \mu (x_1 - 1)$$

$$x_1^2 - (2\mu + 1) x_1 + \mu = 0$$

$$x_1 = \frac{2\mu + 1 \pm \sqrt{(2\mu+1)^2 - 4\mu}}{2}$$

siden $x_1 \ge 1$: $x_1 = \dfrac{2\mu + 1 + \sqrt{4\mu^2 + 1}}{2}$

$$x_2 = \frac{\mu}{x_1 - 1} = \frac{\mu}{(2\mu - 1) + \sqrt{4\mu^2 + 1}} = \frac{2\mu\left(2\mu - 1 - \sqrt{4\mu^2+1}\right)}{(2\mu-1)^2 - (4\mu^2+1)}$$

$$= \frac{2\mu\left(2\mu+1-\sqrt{4\mu^2+1}\right)}{-4\mu}$$

$$= -\frac{-2\mu+1+\sqrt{4\mu^2+1}}{2}$$

$$\vec{x}(\mu) = (x_1, x_2) = \left(\frac{2\mu+1+\sqrt{4\mu^2+1}}{2}, \frac{-2\mu+1+\sqrt{4\mu^2+1}}{2} \right)$$

$\downarrow$

$(1,1)$ når $\mu \to 0$

$$\vec{y}(\mu) = \vec{x}(\mu), \quad \vec{z}(\mu) = \left(\frac{\mu}{x_1(\mu)}, \frac{\mu}{x_2(\mu)} \right) = \vec{w}(\mu)$$

Ex. 17.2

$$\begin{aligned} \max \quad & (\cos\theta)\, y_1 + (\sin\theta)\, x_2 \\ \text{s.t.} \quad & x_1 \leq 1 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$0 \leq \theta \leq \frac{\pi}{2}$

optimum: $(1,1)$

$Ax + w = b$

$A^T y - z = c$

$$\begin{aligned} x_1 \quad + w_1 \quad &= 1 \\ x_2 \quad + w_2 \quad &= 1 \\ y_1 \quad - z_1 \quad &= \cos\theta \\ y_2 \quad - z_2 \quad &= \sin\theta \end{aligned}$$

splitter ulikhetene i to kategorier, indeks 1, indeks 2.

(1)

$x_1 + w_1 = 1$

$y_1 - z_1 = \cos\theta$

$x_1 z_1 = y_1 w_1 = \mu$

(2)

$x_2 + w_2 = 1$

$y_2 - z_2 = \sin\theta$

$x_2 z_2 = y_2 w_2 = \mu$

$w_1 = 1 - x_1$

$$\frac{\mu}{w_1} - \frac{\mu}{x_1} = \cos\theta$$

$$\frac{\mu}{1-x_1} - \frac{\mu}{x_1} = \cos\theta$$

$\vdots$

$7x_1^2 + (2-7)x_1 - 1 = 0$, der $7 = \frac{\cos\theta}{\mu} = \frac{c}{\mu}$

$$x_1 = \frac{\mu - 2 \pm \sqrt{(2-\mu)^2 + 4\mu}}{2\mu} = \frac{\mu - 2 \pm \sqrt{\mu^2 + 4}}{2\mu}$$

since $x_1 \geq 0$

$$= \frac{\mu - 2 + \sqrt{\mu^2 + 4}}{2\mu}$$

$$\vdots$$

$$= \frac{c - 2\mu + \sqrt{c^2 + 4\mu^2}}{2c}$$

Samme for (2): $x_2 = \dfrac{s - 2\mu + \sqrt{s^2 + 4\mu^2}}{2s}$

$$\vec{x}(\mu) = (x_1, x_2) = \left(\frac{c - 2\mu + \sqrt{c^2 + 4\mu^2}}{2c}, \frac{s - 2\mu + \sqrt{s^2 + 4\mu}}{2s} \right)$$

Andre variable: $\vec{w}(\mu) = 1 - \vec{x}(\mu)$

$$\vec{z}(\mu) = \frac{\mu}{\vec{x}(\mu)}$$

$$\vec{y}(\mu) = \frac{\mu}{\vec{w}(\mu)}$$

$\mu \to 0^+$: $\vec{x}(\mu) \to (1, 1)$

$\mu \to \infty$: $\vec{x}(\mu) \to (\frac{1}{2}, \frac{1}{2})$ $\left(\sqrt{\cos^2\theta + 4\mu^2} - 2\mu \right.$

$$\left. = \frac{\cos^2\theta}{\sqrt{\cos^2\theta + 4\mu^2} + 2\mu} \to 0 \right)$$

Ex. 17.3

Med $\vec{r} \in \mathbb{R}^n$, $\vec{s} \in \mathbb{R}^m$ definer,

$$L(\vec{x}, \vec{w}, \vec{y}) = \vec{c}^T \vec{x} + \sum_{j=1}^{n} r_j \log x_j + \sum_{i=1}^{m} s_i \log w_i + \vec{y}^T (b - A\vec{x} - \vec{w})$$

Vi får:

$$\frac{\partial L}{\partial x_j} = c_j + \frac{r_j}{x_j} - (A^T y)_j = 0 \quad \Rightarrow \quad A^T y - \underbrace{X^{-1}\vec{r}}_{\vec{z}} = \vec{c}$$

$$\frac{\partial L}{\partial w_i} = \frac{s_i}{w_i} - y_i = 0 \quad \Rightarrow \quad W^{-1}\vec{s} = \vec{y}$$

$$\frac{\partial L}{\partial y_i} = (b - Ax - w)_i = 0 \quad \Rightarrow \quad A\vec{x} + \vec{w} = \vec{b}$$

Vi skriver om: $A^T\vec{y} - \vec{z} = \vec{c}$

$A\vec{x} + \vec{w} = \vec{b}$

$X\vec{z} = \vec{r}$ $\}$ $XZ\vec{e} = \vec{r}$

$W\vec{y} = \vec{s}$ $\rightarrow$ $x_j z_j = r_j$, $w_i y_i = s_i$

$WY\vec{e} = \vec{r}$

Vi har også: $\frac{\partial^2 f}{\partial x_j^2} = -\frac{r_i}{x_j^2} < 0$, $\frac{\partial^2 f}{\partial w_i^2} = -\frac{s_i}{w_i^2} < 0$,

og alle blandede partielle deriverte er 0.

Derfor er Hf diagonal, med negative diagonalelementer.

Fra kommentar etter teorem 17.1: Et kritisk punkt er unikt hvis det finnes, og er et globalt maks.

Siden "feasible set" er begrenset ved antagelse, så er f begrenset ovenfra, så den har et lokalt maks.

Dette er et kritisk punkt, og må derfor være et maksimum.

Ex. 17.4

(17.8):

(P)

$\max \vec{c}^T\vec{x}$

s.t. $A\vec{x} \leq \vec{0}$

$\vec{x} \geq \vec{0}$

(D)

$\min \vec{b}^T\vec{y}$

s.t. $A^T\vec{y} \geq \vec{c}$

$\vec{y} \geq \vec{0}$

La $\vec{x}$ være basic feasible solution for (P), med $w_1 = w_2 = \dots = w_m = 0$ (i.e. $x \in V$)

La $\vec{y}, \vec{z}$ være tilsvarende basic dual solutions / slacks.

På grunn av komplementær slack: $x_j z_j = x_j(\vec{y}^T\vec{a}_j - c_j) = 0 \ \forall j$

(17.9):

$\max \sum_{j=1}^{n} c_j \xi_j^2$

s.t. $\sum_{j=1}^{n} \vec{a}_j \xi_j^2 = \vec{b}$

(W er settet av punkter som tilfredsstiller førsteordens betingelsene for dette problemet!)

Første ordens betingelser: $L(\xi, y) = \sum_{j=1}^{n} c_j \xi^2 + \vec{y}^T \left(\vec{b} - A \begin{bmatrix} \xi_1^2 \\ \vdots \\ \xi_n^2 \end{bmatrix}\right)$

$$\frac{\partial L}{\partial \xi_j} = 2 c_j \xi_j - 2 \vec{y}^T \vec{a}_j \xi_j = 0$$

$$\frac{\partial L}{\partial y_i} = \left(\vec{b} - A \begin{bmatrix} \xi_1^2 \\ \vdots \\ \xi_n^2 \end{bmatrix}\right)_i = 0$$

Så, førsteordensbetingelsene blir:

$$\xi_j (c_j - \vec{y}^T \vec{a}_j) = 0 \quad \forall j \qquad \vec{x} = \begin{bmatrix} \xi_1^2 \\ \vdots \\ \xi_n^2 \end{bmatrix}$$

$$A \begin{bmatrix} \xi_1^2 \\ \vdots \\ \xi_n^2 \end{bmatrix} = \vec{b}$$

$\longrightarrow$

$$x_j (c_j - \vec{y}^T \vec{a}_j) = 0 \quad \forall j$$

$$A\vec{x} = \vec{b}$$

$V \subset W$

Derfor, $\vec{x} \in V$ impliserer at, for alle $\xi_1, \ldots, \xi_n$, så $\vec{x} = \begin{bmatrix} \xi_1^2 \\ \vdots \\ \xi_n^2 \end{bmatrix}$ tilfredsstille ~~førsteordens~~ betingelsene.