More on pure strategies (not directly in the book)

Recall that a **pure strategy** is one where the player always plays a fixed choice

Define: $P_R(i) = \max_j a_{ij}$ (best pure strategy for K, when known that R uses pure strategy i)

$P_K(j) = \min_i a_{ij}$ (best pure strategy for R, when known that K uses pure strategy j)

$V_* = \max_j P_K(j) = \max_j \min_i a_{ij}$

$V^* = \min_i P_R(i) = \min_i \max_j a_{ij}$

} best pure strategies for K, R when known that the other knows his strategy and exploits that.

If $V_* = V^*$, their common value is called the **value** of the game (V).

If $P_R(r) = V^*$ r

$P_K(s) = V_*$ s

are called pure $\begin{matrix}\text{min max}\\ \text{max min}\end{matrix}$ strategies.

Note: paper/scissors/rock does not have a value:

$P_R(i) = \max_j a_{ij} = \max_j \{-1,0,1\} = 1 \ \forall i \Rightarrow V^* = 1$

$P_K(j) = \min_i a_{ij} = \min_j \{-1,0,1\} = -1 \ \forall i \Rightarrow V_* = -1$

Example: Let $A = \begin{bmatrix} 5 & 2 & 7 & 6 \\ 1 & 2 & 2 & 0 \\ 1 & 4 & 3 & 3 \end{bmatrix}$ (max: $7 = P_R(1)$, $2 = P_R(2)$, $4 = P_R(3)$; min: $P_K(1)=1$, $P_K(2)=2$, $P_K(2)=2$)

$V^* = \min\{7,2,4\} = 2$

$P_k(4) = 0$

$V_* = \max(1, 2, 2, 0) = 2$

Since $V_* = V^*$, the game has value $V = 2$.

Since $P_k(3) = 2$: $j = 3$ is a pure max min strategy

Since $P_R(2) = 2$; $i = 2$ is a pure minmax strategy.

Proposition:

1) $P_k(j) \le a_{ij} \le P_R(i)$

2) $P_k(j) \le V_* \le V^* \le P_R(i)$

$\forall\, i, j$

Proof:

1): $P_k(j) = \min_k a_{kj} \le a_{ij} \le \max_k a_{ik} = P_R(i)$

(def.) (def.)

2) $P_k(j) \le \max_k P_k(k) = V_*$

$P_R(i) \ge \min_k P_R(k) = V^*$

This gives $P_k(j) \le V_*$, $V^* \le P_R(i)$, so first and third inequality in 2) is proved

Also: Take max over j in 1):

$$V_* \le P_R(i)$$

Then take min over i:

$$V_* \le V^*$$

■

A pair (r, s) of pure strategies for R and K is called a **saddle point** if $a_{rj} \le a_{rs} \le a_{is}$ for all i, j

Note: This means that a_{rs} is smallest in columns

a_{rs} is biggest in row r

Example: Back to $\begin{bmatrix} 5 & 2 & 7 & 6 \\ 1 & 2 & 2 & 0 \\ 1 & 4 & 3 & 3 \end{bmatrix}$ (2,2) and (2,3) are saddle points.

Theorem

A game has a value with r as pure minmax strategies for R
s maxmin k

$\Updownarrow$

(r,s) is a saddle point

If these hold: The value of the game is $V = a_{rs}$

Proof: $\Downarrow$ Assume V is the value of the game, and r are pure minmax strategies for R
s maxmin k.

For any i, j: $a_{is} \overset{\text{def.}}{\geq} P_k(s) = V_* = V = V^* = P_R(r) \overset{\text{def.}}{\geq} a_{rj}$

From pure strategy assumptions.

setting $i=r, j=s$ $\quad a_{rs} \geq V \geq a_{rs} \Rightarrow V = a_{rs}$

$\Downarrow$

$a_{is} \geq a_{rs} \geq a_{rj}$, $\Rightarrow (r,s)$ saddle point.

$\Uparrow$: If (r,s) is a saddle point, $a_{rj} \leq a_{rs} \leq a_{is}$ $\forall i, j$

We get

$$V_* \overset{\text{det}}{=} \max_j P_k(j) \geq P_k(s) \overset{\text{def.}}{=} \min_i a_{ij} = a_{rs} \quad \Big\} V^* \leq a_{rs} \leq V_*$$

$$V^* = \min_i P_R(i) \leq P_R(r) \overset{\text{def.}}{=} \max_j a_{rj} = a_{rs} \quad \Big\}$$

since also $V_* \leq V^*$, we have $V = V^* = V_* = a_{rs}$, so that game has value a_{rs}.

Since also $P_k(s) = P_R(r) = V$, r are pure minmax strategies for k
s maxmin R

Ex. 11.5 When are pure strategies simultaneously optimal?

(*) $\begin{matrix} r \\ s \end{matrix}$ pure strategy of $\left.\begin{matrix} R: \vec{y} = \vec{e}_r \\ K: \vec{x} = \vec{e}_s \end{matrix}\right\}$ chooses one item.

$\begin{matrix} k \\ R \end{matrix}$ chooses $\begin{matrix} \vec{x} \\ \vec{y} \end{matrix}$ by $\begin{matrix} \max\limits_x \vec{e}_r A \vec{x} = \max\limits_x (A\vec{x})_r \\ \min\limits_y \vec{y}^T A \vec{e}_s = \min\limits_y (\vec{y}^T A)_s \end{matrix}$

When is $\max\limits_x (A\vec{x})_r = \min\limits_y (\vec{y}^T A)_s$ (i.e., both optimal).

$\max\limits_k a_{rk}$ — max in row r

$\min\limits_c a_{cs}$ — min in column s

Since $\min\limits_c a_{cs} \le a_{rs} \le \max\limits_k a_{rk}$, we must have that

$a_{rs} = \max\limits_k a_{rk} = \min\limits_c a_{cs}$, so a_{rs} = max in row r, and min in column s (i.e. a saddle point)