Exam 2018, problem 2

a)

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -3 & 2 \\ 1 & -2 & -2 \end{pmatrix}$$

max: $\to 2$, $\to 4$, $\to 1$

min: 1, -3, -2

$\min = V^* = 1$

R has pure minmax strategy of 3.

Solution:

$\max = 1 = V_*$ $\Rightarrow$ K has pure maxmin strategy 1

Since $V_* = V^* = 1$, the game has value 1.

(3, 1) is a saddle point

(3: biggest in row; 1: smallest in column.)

b) row / column player wins: sum of numbers is odd / even

$R=1, K=2$: sum is 3, row player wins $\Rightarrow -3$ is in $(1,2)$-entry in A.

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}$$

Saddle points: entries which are biggest in its row, smallest in its column.

$(1,1)$: not saddle: not smallest in column

$(1,2)$: not saddle: not biggest in row

$(2,1)$: not saddle: not biggest in row

$(2,2)$: not saddle: not smallest in column

Therefore: no saddle points!

c) We set $\vec{y} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$, $\vec{x} = \begin{pmatrix} q \\ 1-q \end{pmatrix}$ in the game from b)

expected payoff:

$$\vec{y}^T A \vec{x} = [p \;\; 1-p] \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \vec{x} = [5p-3 \;\; -7p+4] \vec{x}$$

$\vec{x} = \vec{e}_1$: $5p-3$

$\vec{x} = \vec{e}_2$: $-7p+4$

"Equalising strategies" here means that $5p-3 = -7p+4$.

$$\Leftrightarrow \quad 12p = 7 \quad \Leftrightarrow \quad p = \frac{7}{12}$$

average payoff: $5p-3 = 5 \cdot \frac{7}{12} - 3 = \frac{35-36}{12} = -\frac{1}{12}$ (payment to R)

expected payoff: $y^T A x = y^T \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = y^T \begin{bmatrix} 5q-3 \\ -7q+4 \end{bmatrix}$

$\vec{y} = \vec{e}_1$: $5q-3$

$\vec{y} = \vec{e}_2$: $-7q+4$

average payoff: $5q-3 = -7q+4 \Rightarrow q = \frac{7}{12}$

Summary: For $y^* = \begin{pmatrix} p \\ 1-p \end{pmatrix}$, $\max_x y^T A x = -\frac{1}{12}$ for $p = \frac{7}{12}$

For $x^* = \begin{pmatrix} q \\ 1-q \end{pmatrix}$, $\min_y y^T A x = -\frac{1}{12}$ for $q = \frac{7}{12}$

Is the game fair? no since $-\frac{1}{12} \neq 0$.

A mini-introduction to convexity.

- $C \subseteq \mathbb{R}^n$ is called convex if $(1-\lambda)\vec{x}_1 + \lambda \vec{x}_2 \in C$ when $\vec{x}_1, \vec{x}_2 \in C$, $\lambda \in [0,1]$ (line segment from $\vec{x}_1$ to $\vec{x}_2$)
- A function f defined on a convex set C is called convex if

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda) f(x_1) + \lambda f(x_2)$$

(inner argument convex)

whenever $x_1, x_2 \in C$, $\lambda \in [0,1]$

Why convexity?

1. Used in optimization. If f is defined on $A \subseteq \mathbb{R}^n$,

a $\begin{matrix}\text{global}\\ \text{local}\end{matrix}$ minimum of f is a point x^* s.t. $f(x^*) \leq f(x)$ $\begin{matrix}\text{all } x \in A\\ x \text{ close to } x^*\end{matrix}$

If f is convex, a local minimum is also a global minimum!

2. Used in approximation

Given some closed set $S \subseteq \mathbb{R}^n$, and $a \notin S$, find a nearest point x to a from S, i.e. x should satisfy

$$\|x-a\| \leq \|y-a\| \quad \text{for all } y \in S.$$

S nonempty, closed: There will always be a nearest point.
If S is also convex: the nearest point is also unique!

Terminology:

- For $\lambda \in [0,1]$, $(1-\lambda)x_1 + \lambda x_2$ is called a convex combination of x_1 and x_2

So: S is convex $\Leftrightarrow$ S is closed under convex combinations

In general: $\sum_{j=1}^{n} \lambda_j x_j$ is called a convex combination when $\lambda_j \geq 0$, $\sum_{j=1}^{n} \lambda_j = 1$

- A polyhedron $P \subseteq \mathbb{R}^n$ is a set on the form $P = \{x \in \mathbb{R}^n : A\vec{x} \leq \vec{b}\}$,

i.e., an intersection of half planes.

$\{x \in \mathbb{R}^n : A\vec{x} = \vec{b}\}$ is also a polyhedron (rewrite each equality in terms of inequalities)

Proposition 1: Polyhedras are convex

Proof: Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron, and let $x_1, x_2 \in P$

$$A((1-\lambda)x_1 + \lambda x_2) = (1-\lambda)Ax_1 + \lambda Ax_2 \leq (1-\lambda)b + \lambda b = b,$$

so that $(1-\lambda)x_1 + \lambda x_2 \in P$ also, so that P is convex. ∎

Linear optimisation and convexity

For the LP problem

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \\ & x \ge 0 \end{aligned}$$

, the feasible set $\{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$ is a polyhedron ($x \ge 0 \Leftrightarrow -x \le 0$)

The feasible set is thus convex

LP: maximising linear functions over polyhedra.

Proposition 2: If an LP problem has a finite optimal value, then the set P^* of optimal solutions is a polyhedron, so also convex

Proof: Let v^* be the optimal value. We have

$$P^* = \{ x \in \mathbb{R}^n : Ax \le b, x \ge 0, c^T x = v^* \}$$

This is a polyhedron ∎

Convex hull: If S is a general set, what is the smallest convex set containing it?

Define $\underbrace{\text{conv}(S)}_{\text{convex hull}}$ as the set of all convex combinations of points in S.

$\text{conv}(S)$ is convex:

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = \lambda_1 x_1 + (1-\lambda_1)\left(\frac{\lambda_2}{1-\lambda_1} x_2 + \frac{\lambda_3}{1-\lambda_1} x_3\right) \quad (\lambda_1+\lambda_2+\lambda_3 = 1)$$

Proposition 3 $\text{conv}(S)$ equals the intersection of all convex sets containing S

Proof: $S \subseteq \text{conv}(S)$ (take combinations with $x_1 = x_2$)

Let W be the intersection of all convex sets containing S.

We have $W \subseteq \text{conv}(S)$

convex set containing S.

Also, a convex set containing S must also contain $\text{conv}(S)$:

$$\text{conv}(S) \subseteq W$$

It follows that $W = \text{conv}(S)$ ∎

If S is a finite set, $\text{conv}(S)$ is also called a polytope.

Example: If $P = \text{conv}\{x_1, \ldots, x_t\}$ is a polytope, solve $\max_{x \in P} c^T x$.

Solution: Write a feasible x as $x = \sum_{j=1}^{t} \lambda_j x_j \quad (\sum_{j=1}^{t} \lambda_j = 1)$

Define $v^* = \max_j c^T x_j$. We have

$$c^T x = c^T\left(\sum_{j=1}^{t} \lambda_j x_j\right) = \sum_{j=1}^{t} \lambda_j c^T x_j \le \sum_{j=1}^{t} \lambda_j v^* = v^*$$

Since $c^T x_j = v^*$ for some j, it follows that v^* is the optimal value, and is obtained for x so that $\lambda_j = 0$ when $c^T x_j < v^*$

Let $J = \{ j : c^T x_j = v^* \}$. The optimal solutions are $\text{conv}(\{x_j\}_{j \in J})$ ∎