When is a system of linear equations consistent?

Lemma 4 Farkas lemma Let A be an $m \times n$ matrix. Then

1. $Ax \leq b$ has at least one solution

$\Updownarrow$

2. $y^T b \geq 0$ for every $y \in \mathbb{R}^m$ satisfying $y^T A = 0, y \geq 0$

Think of $\vec{y}$ as values used to combine the constraints. statement 2 then says that we can't produce an inequality on the form $0 \leq -1$.

Proof:

(P)

$\max c^T x$

s.t. $Ax \leq b$

$\longleftrightarrow$

(D)

$\min b^T y$

s.t. $A^T y = c$

$y \geq 0$

Let $\vec{c} = 0$, (D) is then feasible ($y = 0$)

By LP-theory: (D) is either unbounded, or has an optimal value.

1. (D) is unbounded $\Leftrightarrow$ (P) infeasible $\Leftrightarrow$ $Ax \leq b$ has no solution.

2. (D) has an optimal value $\Leftrightarrow$ $Ax \leq b$ has a solution $\Leftrightarrow$ $y^T b \geq 0$ for every $y \in \mathbb{R}^m$ so that $y^T A = 0, y \geq 0$

■

Farkas lemma geometrically:

Another version:

(*) $Ax = b$ has a solution $x \geq 0$

(**) $y^T b \geq 0$ for all y with $y^T A \geq 0$.

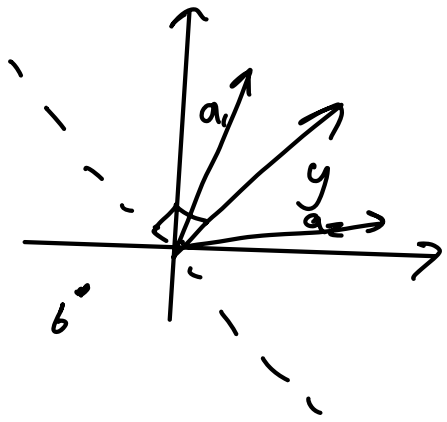
Let $a_1, a_2, \ldots, a_n$ be the columns in A, C the cone they generate

nonnegative linear combinations of $a_1, \ldots, a_n$

By Farkas lemma (*): $b \in C$

the opposite of (**): $\exists y \mid y^T A \geq 0,\ y^T b < 0$

$\exists y \mid y^T a_1 \geq 0, \dots, y^T a_n \geq 0,\ y^T b < 0$



y is a normal vector of a hyperplane ($y^T x = 0$) that separates b from $a_1, \dots, a_n$.

(**) b and the $\{a_i\}$ can not be separated by a hyperplane.

Main theorem for polyhedra

Theorem 5: Let $P \subseteq \mathbb{R}^n$ be a nonempty polyhedron

Then there are vectors $v_1, \dots, v_s, w_1, \dots, w_t \in \mathbb{R}^n$ so that

(*) $P = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{s} \lambda_i v_i + \sum_{j=1}^{t} \mu_j w_j,\ \lambda_i, \mu_j \geq 0,\ \sum \lambda_i = 1\}$

Moreover: for given vectors v_i, w_j, the set of vectors

(*) form a polyhedron

$Ax \leq b$: Exterior description.

(*) Interior

corollary 6: P is a polytope $\Leftrightarrow$ P is a bounded polyhedron

Connection to LP:

$$\max c^T x \text{ s.t. } Ax \leq b \iff x = \sum_{i=1}^{s} \lambda_i v_i + \sum_{j=1}^{t} \mu_j w_j$$

Two possibilities:

- $c^T w_j > 0$ for some j: the problem is unbounded (set $x = \alpha w_j, \alpha \to \infty$)
- $c^T w_j \leq 0$ for all j: The problem has an optimal solution as one of the v_i's.

The v_i's can be chosen as extreme points of P

points in P that cannot be written as convex combinations of other points in P: $x = \frac{1}{2}x' + \frac{1}{2}x^2 \Rightarrow x = x' = x^2$.

proposition 7 Let A be $m \times n$ with rank $A = m$, and define

$$P = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}$$

Let $x \in P$. Then: x is a basic solution in the LP sense $\Leftrightarrow$ x is an extreme point in P.

Proof: $\Rightarrow$ Assume x is a basic solution. Write $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$ $(Ax = Bx_B + Nx_N = b)$

$x_B = B^{-1}b$

Assume that $x = \frac{1}{2}x' + \frac{1}{2}x^2$ for $x', x^2 \in P$

We need to show that $x = x' = x^2$, so that x is extreme point.

We have $x_B = \frac{1}{2}x_B' + \frac{1}{2}x_B^2$, $x_N = \frac{1}{2}x_N' + \frac{1}{2}x_N^2$

$x_N = 0 \Rightarrow x_N' = x_N^2 = 0$ (since both $x_N', x_N^2 \geq 0$)

$Ax' = b \Rightarrow Bx_B' + N\underbrace{x_N'}_{0} = b \Rightarrow x_B' = B^{-1}b = x_B$

since $x' \in P$

Similarly $x_B^2 = x_B$. Since $x_B = x_B' = x_B^2$, $x_N = x_N' = x_N^2 = 0$

$x = x' = x^2$.

$\Leftarrow$ Assume that x is an extreme point. Set $J = \{ j : x_j > 0 \}$

Claim: The columns of A with indices ~~from~~ J are linearly independent

Proof: Let a_j be column j in A, and assume that $\sum_{j \in J} \gamma_j a_j = 0$

Then also $\sum_{j \in J} \varepsilon \gamma_j a_j = 0$ for any ε.

$$Ax = b \Rightarrow \sum_{j \in J} x_j a_j = b \Rightarrow \sum_{j \in J} (x_j \pm \varepsilon \gamma_j) a_j = b$$

choose ε so that this is > 0 for all $j \in J$ possible since $x_j > 0$ for $j \in J$.

Define x^1, x^2 by: $x_1^j = \begin{cases} x_j + \varepsilon \lambda_j & j \in J \\ 0 & j \notin J \end{cases}$ $x_2^j = \begin{cases} x_j - \varepsilon \lambda_j & j \in J \\ 0 & j \notin J \end{cases}$

Then $x^1, x^2 \in P$ (since $Ax^1 = b$, $Ax^2 = b$, and $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$

Since x is extreme points, $x = x^1 = x^2$, so that $\varepsilon \lambda_j = 0 \Rightarrow \lambda_j = 0$

$\Rightarrow$ linearly independent ■

Extend the columns $\{a_j\}_{j \in J}$ to a basis for $\mathbb{R}^m$ by including other columns from A (possible since rank $A = m$), and let B be the corresponding basis.

We have that $x_N = 0$ (since $x_j > 0$ for $j \in J$ only)

But then $Bx_B + Nx_N = Bx_B = b \Rightarrow x_B = B^{-1}b$,

so that x is the basic solution corresponding to B ■

Example 5

(i) Let a_i be scalars and set $P = \{x \in \mathbb{R}^n : a_1 x_1 + \cdots + a_n x_n = b, x_i \geq 0\}$

$A = [a_1 \cdots a_n]$

A basis is here simply an entry in A, and B is a 1×1 submatrix.

B is invertible $\Leftrightarrow a_j \neq 0$

basic solution: $x_B = B^{-1}b = \frac{b}{a_j} \Rightarrow x = (0, \ldots, 0, \frac{b}{a_j}, 0, \ldots, 0)$

in entry j.

(ii) $P = \left\{ x \in \mathbb{R}^n : \begin{array}{rl} 2x_1 + 3x_2 & = 12 \\ x_1 + x_2 + x_3 + x_4 & = 1 \\ x_i & \geq 0 \end{array} \right.$

Here $A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, which has rank $2 = m$

possible bases: $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}$.

(iii) $P = \{x \in \mathbb{R}^n : \sum_{j=1}^n a_j x_j \le b,\ 0 \le x \le e\}$ (e: vector of all ones) $(a_j, b > 0)$

extreme points:

1/ if $\sum_{j=1}^n a_j x_j < b$, and $0 < x_j < 1$ for some j:

Set $x = \frac{1}{2}(x + \varepsilon_j e_j) + \frac{1}{2}(x - \varepsilon_j e_j)$, so x is not extreme point.

inf for ε small.

Thus, candidates only for $x_j \in \{0, 1\}$.

2/ if $\sum_{j=1}^n a_j x_j = b$: Assume two j's satisfy $0 < x_j < 1$.

Increase one of the x_j and decrease the other, so that still $0 < x_j < 1$, and still $\sum_{j=1}^n a_j x_j = b$, to obtain a decomposition

$$x = \frac{1}{2}x' + \frac{1}{2}x^2,$$

so that x is not an extreme point.

Thus, only one x_j can satisfy $0 < x_j < 1$, and new candidates are easily found from $\sum_{j=1}^n a_j x_j = b$

in (iii): the matrix A can be found by adding slacks:

$$\left.\begin{aligned} \textstyle\sum a_j x_j + w_0 &= b \\ x_j + w_j &= 1 \end{aligned}\right\} \quad A = \begin{pmatrix} a_1 \cdots a_n & 1 & & \\ 1 & 0 \cdots 0 & 1 & \\ & 1 \quad 0 & & 1 \\ 0 & 0 \cdots & 1 & 0 \quad 1 \\ & & \vdots & \end{pmatrix}$$

$$b = \begin{pmatrix} b \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$