Fourier-Motzkin elimination

Eliminating variables in $Ax \le b$ as in Gaussian elimination.

set $I_+ = \{ i : a_{i1} > 0 \}$

$I_0 = \{ i : a_{i1} = 0 \}$

$I_- = \{ i : a_{i1} < 0 \}$

$Ax \le b$ can be split into:

$i \in I_+$: $x_1 \le \frac{1}{a_{i1}} (b_i - \sum_{k>1} a_{ik} x_k) \Leftarrow a_{i1} x_1 + \cdots + a_{in} x_n \le b$

$i \in I_0$: $(0 \cdot x_1 + \cdots + a_{in} x_n = b)$

$i \in I_-$: $x_1 \ge \frac{1}{a_{i1}} (b_i - \sum_{k>1} a_{ik} x_k)$

This is equivalent to:

$$\underbrace{\frac{1}{a_{j1}}}_{\text{neg}} (b_j - \sum_{k>1} a_{jk} x_k) \le x_1 \le \underbrace{\frac{1}{a_{i1}}}_{\text{pos}} (b_i - \sum_{k>1} a_{ik} x_k) \qquad \begin{matrix} i \in I_+ \\ j \in I_- \end{matrix}$$

also $0 \cdot x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \le b_i$ for $i \in I_0$

Substitute $\frac{b_j}{a_{j1}} \to -b_j' \quad j \in I_-$

$\frac{b_i}{a_{i1}} \to b_i' \quad i \in I_+$

similarly $a_{ij} \to a_{ij}'$

We get $\sum a_{jk}' x_k - b_j' \le x_1 \le b_i' - \sum_{k>1} a_{ik}' x_k \qquad \begin{matrix} i \in I_+ \\ , j \in I_- \end{matrix}$

where a_{jk}', b_j' have the same signs as a_{jk}, b_j.

Thus: $x_1, \ldots, x_n$ solve $Ax \le b$

$\Updownarrow$

$$\max_{j \in I_-} \left(\sum a_{jk}' x_k - b_j' \right) \le x_1 \le \min_{i \in I_+} \left(b_i' - \sum_{k>1} a_{ik}' x_k \right)$$

$a_{i2} x_2 + \cdots + a_{in} x_n \le b_i \quad (i \in I_0)$

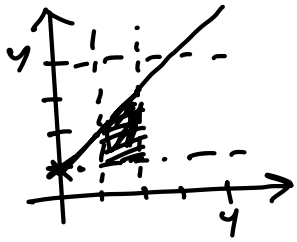
Strategy: Solve these $|I_+| \cdot |I_-| + |I_0|$ inequalities, by reducing to $n-2$ variables and so on.

Finally we end up with one variable $c \le x_n \le d$.

General solution: Choose x_n so that $c \le x_n \le d$,
choose x_{n-1} from an interval depending on x_n,
choose x_{n-2} from an interval depending on x_n, x_{n-1},
and so on

Theorem 8 Fourier-Motzkin elimination is a finite algorithm that finds a general solution to $Ax \le b$.
If no solution: The algorithm produces an inconsistent inequality $c \le d$

Example 6: Consider $1 \le x_1 \le 2$
$1 \le x_2 \le 4$
$x_1 - x_2 \ge -1$



We have
I_- : $x_1 \ge 1$, $x_1 \ge x_2 - 1$ (2 ineq.)
I_+ : $x_1 \le 2$ (1 ineq.)
I_0 : $1 \le x_2 \le 4$ (2 ineq.)

$$\underbrace{\max(1, x_2 - 1)}_{I_-} \le x_1 \le \underbrace{\min(2)}_{I_+}, \quad \underbrace{1 \le x_2 \le 4}_{I_0}$$

$$1 \le 2, \quad x_2 - 1 \le 2, \quad 1 \le x_2 \le 4$$

$$x_2 \le 3, \quad 1 \le x_2, \quad x_2 \le 4$$

$$1 \le x_2 \le 3 \iff x_2 \in [1, 3].$$

Thus: solution is: $1 \le x_2 \le 3, \quad \max(1, x_2 - 1) \le x_1 \le 2$

Exercise 1 $B(a,r) = \{x \in \mathbb{R}^n : \|x-a\| \le r\}$ is convex:

Suppose $x, y \in B(a,r)$, so that $\|x-a\| \le r$, $\|y-a\| \le r$

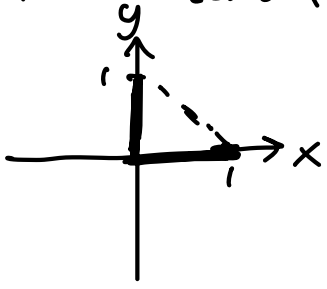
$$\|(1-\lambda)x + \lambda y - a\| = \|(1-\lambda)(x-a) + \lambda(y-a)\|$$
$$\le \|(1-\lambda)(x-a)\| + \|\lambda(y-a)\|$$
$$= (1-\lambda)\|x-a\| + \lambda\|y-a\| \le (1-\lambda)r + \lambda r = r$$

$\Rightarrow (1-\lambda)x + \lambda y \in B(a,r)$, so that $B(a,r)$ is convex.

Exercise 2 All vector spaces V are convex:

(all convex combinations are also linear combinations)

Exercise 3 Union of two convex sets may not be convex:



Exercise 4 $C_1 \cap \cdots \cap C_t$ is convex when $C_1, \ldots, C_t$ are:

Let $x, y \in C_1 \cap \cdots \cap C_t$. Then $x, y \in C_i$ for all i, so that $(1-\lambda)x + \lambda y \in C_i$ for all i (since C_i convex), so that $(1-\lambda)x + \lambda y \in C_1 \cap \cdots \cap C_t \Rightarrow C_1 \cap \cdots \cap C_6$ is convex.

Exercise 5 $B(0,1)$ is not a polyhedron:

Assume that $\{x \in \mathbb{R}^n : \|x\|_2 \le 1\} = \{x \in \mathbb{R}^n : Ax \le b\}$ for some A, b.

Let $a^T x \le \alpha$ be an inequality in $Ax \le b$

We can assume that $a^T x = \alpha$ is tangent to the sphere at some point b with $\|b\|_2 = 1$

$\downarrow$

$b^T x \le 1$

So, we assume all inequalities are on the form $a_i^T x \leq 1$, $\|a\|_2 = 1$

So, assume $B(0,1) = \{ x \in \mathbb{R}^n : a_i^T x \leq 1 \}$

if $x \neq a_i$, $\|x\|_2 = 1$, we have that $a_i^T x < 1$ for all i.

We can choose a point x' so that $\|x'\|_2 > 0$, and so that still $a_i^T x < 1$ for all i

This implies that $x' \in \{ x \in \mathbb{R}^n : a_i^T x \leq 1 \}$, but $x' \notin B(0,1)$

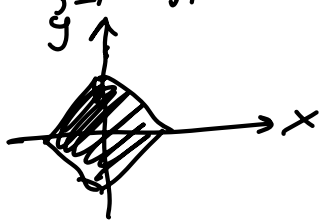
Thus $B(0,1)$ is not a polyhedron.

Exercise 6 $B_\infty = \{ x \in \mathbb{R}^n : \|x\|_\infty \leq 1 \}$ $\|x\|_\infty := \max_j |x_j|$

B_∞ is a polyhedron:

$$B_\infty = \{ x \in \mathbb{R}^n : \|x\|_\infty \leq 1 \}$$
$$= \{ x \in \mathbb{R}^n : |x_j| \leq 1 \ \forall j \}$$
$$= \{ x \in \mathbb{R}^n : -1 \leq x_j \leq 1 \ \forall j \}$$
$$= \{ x \in \mathbb{R}^n : x_j \leq 1, -x_j \leq 1 \ \forall j \}, \text{ which is a polyhedron.}$$

Exercise 7 $B_1 = \{ x \in \mathbb{R}^n : \|x\|_1 \leq 1 \}$ $\|x\|_1 = \sum_{j=1}^n |x_j|$



B_1 is a polyhedron:

$$B_1 = \{ x \in \mathbb{R}^n : \|x\|_1 \leq 1 \}$$
$$= \{ x \in \mathbb{R}^n : |x_1| + \cdots |x_n| \leq 1 \}$$
$$= \{ x \in \mathbb{R}^n : \pm x_1 \pm x_2 \pm \cdots \pm x_n \leq 1 \}$$ (all sign combinations)

2^n inequalities

so that B_1 is a polyhedron.

Exercise 8

$\max c^T x$ s.t. $Ax \le b$: set $x = x^+ - x^-$ $(x^+, x^- \ge 0)$

$Ax \le b \iff Ax + w = b,\ w \ge 0$

$A(x^+ - x^-) + w = b \quad x^+, x^-, w \ge 0$

$$\underbrace{[A \ -A \ I]}_{A_1} \underbrace{\begin{bmatrix} x^+ \\ x^- \\ w \end{bmatrix}}_{x'} = \underbrace{b}_{b_1}$$

$A_1 x_1 = b_1$, $x_1 \ge 0$

$$c^T x = c^T (x^+ - x^-) = \underbrace{\begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}^T}_{c_1} \begin{bmatrix} x^+ \\ x^- \\ w \end{bmatrix}$$

Exercise 9

$P = \{x : 0 \le x_i \le 1\}$

$\max c^T x : x \in P$

make each $c_i x_i$ as big as possible: $c_i > 0$: choose $x_i = 1$ to max.

$c_i \le 0$: choose $x_i = 0$ to max.

maximum value : $\sum_{i : c_i > 0} c_i$

What is $\max_{a_i \le x_i \le b_i} c^T x$?

if $a_i, b_i > 0$ for all i : $\sum_{c_i > 0} c_i b_i + \sum_{c_i \le 0} c_i a_i$